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SECOND-VARIATIONAL CALCULUS OF VARIATIONS METHOD

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MANNED SPACECRAFT CENTER

HOUSTON, TEXAS

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CONTENTS

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Bobby R. Uzzell and Scott S. McKay

1. INTRODUCTION

This paper presents the mathematical methods used in the development of the second-variational calculus of variations optimization programs. An outline of the derivation of these methods for a constant thrust, single stage problem is presented in reference 2. Before the work presented in this paper was performed, the method had been applied only to a single stage, two-dimensional, constant thrust program. This paper presents a fairly detailed discussion of the methods and demonstrates the formulation of the necessary equations for the solution of a three-dimensional insertion into a circular orbit for a rocket engine with variable thrust capability. No results and no program discussions are presented.

It must be emphasized that the work presented in this paper would not have been possible, or at best would have required a considerably longer time for development without the assistance of Dr. H. J. Kelley of Analytical Mechanics Associates, Inc.

2. STATEMENT OF MAYER PROBLEM

Consider a particle whose movement in space can be expressed by a system of first-order differential equations

$$\dot{x}_j = g_j\left(t, x_1, \ldots, x_n, y_1, \ldots, y_m\right), \qquad j = 1, \ldots, n, \tag{1}$$

where

$$\dot{x}_j = \frac{dx_j}{dt}$$

Time, t, represents the independent variable and the functions $x_i(t)$ and $y_k(t)$ are dependent variables. The equations (1) are referred to as the state equations in that they determine the values of the state variables x_i. The system (1) essentially represents a set of constraints that must be met by the state variables as the particle moves in space. The set y_k is termed as the control variable vector $y(t)$ since values can be prescribed as functions of time to control the behavior of the system. For meaningful problems it is required that the system, described by equations (1), move from a prescribed initial state and time to a prescribed terminal time and state. The initial and final states will be in general only partially prescribed. In general, it is required that the solution of state equations (1), with some control $y(t)$, satisfy end conditions

$$\begin{gathered}\psi_r\left[t_o, t_f, x_1\left(t_o\right), \ldots, x_n\left(t_o\right), x_1\left(t_f\right), \ldots, x_n\left(t_f\right)\right] = 0 \\ r < 2n + 2\end{gathered} \tag{2}$$

The problem would be overdetermined if $r \geq 2n + 2$. It is desired that a $y(t)$ is found such that the system defined above will behave in such a manner as to provide a minimum value of a scalar function

$$P\left[t_o, t_f, x_1\left(t_o\right), \ldots, x_n\left(t_o\right), x_1\left(t_f\right), \ldots, x_n\left(t_f\right)\right] \tag{3}$$

The problem posed thus far - namely, that of finding a $y(t)$ which minimizes a function of the terminal values of the state variables of a system described by differential equations subject to prescribed end conditions - is known as the general problem of Mayer with mixed end conditions.

Since the control can be prescribed it is necessary to define the admissible space of control variables. The domain of definition is t. Let U_k be the range of the control variables y_k. U denotes the range of $y(t)$. Given a point,

$$y = \left(y_1, y_2, \ldots, y_m\right)$$

which is an element of U, is equivalent to giving a numerical system of parameters $y_1, y_2, \ldots, y_m$. In applications some y_k may be defined on a closed set. For the discussion in this paper the control variables will be piecewise continuous, that is, control $y = y(t)$ is continuous for all t under consideration, with the exception of only a finite number of t, at which $y(t)$ may have discontinuities of the first kind. For discontinuities of the first kind the following limits exist at a point of discontinuity at $t = \xi$:

$$y_k(\xi - 0) = \lim_{\substack{t \to \xi \\ t < \xi}} y_k(t)^-, \qquad y_k(\xi + 0) = \lim_{\substack{t \to \xi \\ t > \xi}} y_k(t)^+ ,$$

where

$$y_k(\xi - 0)$$

may not be equal to

$$y_k(\xi + 0)$$

The value of a piecewise continuous control $y(t)$ is assumed to have the following values at the point of discontinuity:

$$y_k(\xi) = y_k(\xi - 0)$$

The vector $y(t)$ is assumed to be continuous at the two extreme endpoints. From this definition it follows that every $y(t)$ is bounded even if U is not.

Thus, any piecewise continuous function $y(t)$, $t_o \le t \le t_f$, whose range is in U, whose values are assigned the left-hand value at a point of discontinuity, and which is continuous at the endpoints of the interval $t_o \le t \le t_f$ on which it is given, is an admissible control.

From the definition of an admissible control it follows that the state variables are continuous, but the time derivative of the state variables may be discontinuous when $y(t)$ is discontinuous.

3. EULER AND TRANSVERSALITY CONDITIONS FOR AN EXTREMUM

The following discussion of the necessary conditions for a minimum is not intended to be rigorous but is meant only to supply a general approach to the development of these conditions. Presenting a rigorous discussion would involve a formidable amount of work. This discussion is based on general knowledge like knowing that the first derivative of an algebraic function must be zero at the point the function has a relative minimum.

Consider the Mayer problem as defined in the previous section, only restrict the problem further by fixing all the initial conditions including time. Also assume that a set y_k has been determined that minimizes $P\left(x_{1f}, \ . \ . \ ., x_{nf}, t_f\right)$ where $x_{if} = x_i\left(t_f\right)$. Since the initial values of the state variables are fixed, P is a function of only the final values. Since the set (1) actually represents constraints, the multiplier rule can be employed so that P can be adjoined with differential constraints to obtain

$$J = P\left(x_{1f}, \ . \ . \ ., x_{nf}, t_f\right) + \int_{t_o}^{t_f} \sum_{i=1}^{n} \lambda_i \left(- \dot{x}_i + g_i\right) dt \qquad (4)$$

where λ_i represents a system of Lagrange multipliers which vary as functions of time.

If an expansion is made about (4),

$$J = J_o + J_1 + \frac{1}{2} J_2 + \ . \ . \ .$$

where J_o corresponds to the nominal path, J_1 the first order effect of control variations and initial variations in state, and J_2 the second order effects. Since J_o was defined to be a minimum, J_1 must be equal to zero and it is desired to determine what properties must be satisfied for J_1 to be zero.

Considering only the first order effect of variations,

$$J_1 = \delta P + \int_{t_o}^{t_f} \sum_{i=1}^{n} \lambda_i\left(-\delta\dot{x}_i + \delta g_i\right)dt + \sum_{i=1}^{n} \lambda_{if}\left(-\dot{x}_{if} + g_{if}\right)\delta t_f \tag{5}$$

The last term in the expression is due to t_f being free and subject to variation δt_f. The terms δP and δg_i are given by the following two expressions, respectively

$$\delta P = \frac{\partial P}{\partial t_f}\delta t_f + \sum_{i=1}^{n} \frac{\partial P}{\partial x_{if}}\left(\delta x_{if} + \dot{x}_{if}\,\delta t_f\right)$$

$$\delta g_i = \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j}\delta x_j + \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k}\delta y_k$$

For shorthand notation let

$$\Delta x_{if} = \delta x_{if} + \dot{x}_{if}\,\delta t_f$$

The Hamiltonian, H, is defined as

$$H = \sum_{i=1}^{n} \lambda_i\, g_i$$

J_1 can now be written in the following form:

$$J_1 = \frac{\partial P}{\partial t_f}\delta t_f + \sum_{i=1}^{n} \frac{\partial P}{\partial x_{if}}\Delta x_{if} + \int_{t_o}^{t_f} \sum_{i=1}^{n} \lambda_i\left(-\delta\dot{x}_i + \sum_{j=1}^{n}\frac{\partial g_i}{\partial x_j}\delta x_j + \sum_{k=1}^{m}\frac{\partial g_i}{\partial y_k}\delta y_k\right)dt + H_f\,\delta t_f - \sum_{i=1}^{n}\lambda_{if}\,\dot{x}_{if}\,\delta t_f \tag{6}$$

Rewrite the integral in (6) in the following form:

$$- \int_{t_o}^{t_f} \sum_{i=1}^{n} \lambda_i \, \delta \dot{x}_i \, dt + \int_{t_o}^{t_f} \sum_{i=1}^{n} \lambda_i \left(\sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} \delta x_j + \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k} \delta y_k \right) dt$$

Letting

$$u_i = \lambda_i$$

and

$$dv_i = \delta \dot{x}_i \, dt$$

and integrating the first integral by parts, the expression is changed to

$$- \sum_{i=1}^{n} \lambda_i \, \delta x_i \Big]_{t_o}^{t_f} + \int_{t_o}^{t_f} \sum_{i=1}^{n} \dot{\lambda}_i \, \delta x_i \, dt$$

$$+ \int_{t_o}^{t_f} \sum_{i=1}^{n} \lambda_i \left(\sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} \delta x_j + \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k} \delta y_k \right) dt$$

Substituting the above expression into (6),

$$J_1 = \frac{\partial P}{\partial t_f} \delta t_f + \sum_{i=1}^{n} \frac{\partial P}{\partial x_{if}} \Delta x_{if} - \sum_{i=1}^{n} \lambda_{if} \, \delta x_{if}$$

$$+ \int_{t_o}^{t_f} \left[\sum_{i=1}^{n} \left(\dot{\lambda}_i + \sum_{j=1}^{n} \frac{\partial g_j}{\partial x_i} \lambda_j \right) \delta x_i + \sum_{i=1}^{n} \lambda_i \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k} \delta y_k \right] dt$$

$$+ H_f \, \delta t_f - \sum_{i=1}^{n} \lambda_{if} \, \dot{x}_{if} \, \delta t_f$$

Rearranging terms in the above expressions, the following result is obtained:

$$J_1 = \left(\frac{\partial P}{\partial t_f} + H_f\right) \delta t_f + \sum_{i=1}^{n} \left(\frac{\partial P}{\partial x_{if}} - \lambda_{if}\right) \Delta x_{if}$$

$$+ \int_{t_o}^{t_f} \left[\sum_{i=1}^{n} \left(\dot{\lambda}_i + \sum_{j=1}^{n} \frac{\partial g_j}{\partial x_i} \lambda_j\right) \delta x_i + \sum_{i=1}^{n} \lambda_i \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k} \delta y_k \right] dt$$

For J_1 to be equal to zero the following four expressions must be equal to zero

$$\left(H_f + \frac{\partial P}{\partial t_f}\right) \delta t_f \tag{7}$$

$$\left(\frac{\partial P}{\partial x_{if}} - \lambda_{if}\right) \Delta x_i \left(t_f\right), \qquad i = 1, \ldots, n \tag{8}$$

$$\left(\dot{\lambda}_i + \sum_{j=1}^{n} \frac{\partial g_j}{\partial x_i} \lambda_j\right), \qquad i = 1, \ldots, n \tag{9}$$

$$\sum_{i=1}^{n} \lambda_i \frac{\partial g_i}{\partial y_k}, \qquad k = 1, \ldots, m \tag{10}$$

From equation (7), if final time is not fixed - that is, $\delta t_f = 0$ - the transversality condition, or natural boundary condition, for open final time is determined and is equal to

$$H_f + \frac{\partial P}{\partial t_f} = 0 \tag{11}$$

From equation (8), if the final x_i is not fixed then the condition

$$\frac{\partial P}{\partial x_{if}} - \lambda_{if} = 0 \tag{12}$$

must be satisfied. This condition is the transversality condition for free x_{if} and is particularly important since it assigns a value to λ_{if}.

Since

$$\frac{\partial H}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial g_j}{\partial x_i} \lambda_j$$

equations (9) can be written as

$$\left(\dot{\lambda}_i + \frac{\partial H}{\partial x_i}\right), \qquad i = 1, . . ., n$$

This result yields the conditions that

$$\dot{\lambda}_i + \frac{\partial H}{\partial x_i}, \qquad i = 1, . . ., n \tag{13}$$

must be equal to zero.

Since

$$\frac{\partial H}{\partial y_k} = \sum_{i=1}^{n} \frac{\partial g_i}{\partial y_k} \lambda_i$$

condition (10) can be written as

$$\frac{\partial H}{\partial y_k} = 0, \qquad k = 1, . . ., m \tag{14}$$

Conditions (13) and (14) are known as the Euler conditions and must be satisfied at every point of the trajectory. Conditions (11), (12), (13), and (14) represent necessary conditions that must be satisfied for an extremum of P.

4. WEIERSTRASS NECESSARY CONDITION FOR A MINIMUM

The control variables y_k along an optimal trajectory satisfy the Weierstrass condition

$$H(y_1^*, \ . \ . \ ., y_m^*) \geq H\ (y_1, \ . \ . \ ., y_m)$$

where the y_k^* are arbitrary. This statement simply means that the vector y minimizes H. The set y_k is often determined from the operation

$$\underset{y(t)}{\text{Min.}} \ H.$$

If inequality constraints on the y_k of the form $y_{k1} \leq y_k \leq y_{k2}$ are required, the minimum operation is performed subject to the constraints. This restricted condition is known as the Pontryagin principle. In rocket applications if thrust, T, is a variable, it is considered as a control variable and is permitted to vary between the limits

$$0 \leq T_1 \leq T \leq T_2$$

Since thrust will appear linearly in the system (1), H can be written in the following form:

$$H = T\ f + a$$

According to the Pontryagin principle, when f is negative thrust should be set to T_2 and when f is positive thrust should be set to T_1. When f is equal to zero, or $\frac{\partial H}{\partial T} = 0$, the thrust level is changed. The function f is defined as the switching function. Since $\frac{\partial H}{\partial T}$ is equal to zero at a switching point, H will remain constant and continuous.

A difficulty occurs if f is zero over a set of measure greater than zero since any thrust level would minimize H. This occurence is not expected for the present problems being studied.

5. EXPRESSION FOR CHANGES IN FINAL STATE VALUES AND DEFINITION OF THE PENALTY FUNCTION

a. As discussed in Statement of Mayer Problem, the system of differential equations to be satisfied along the flight path is given in first-order form by system (1). It is now assumed that a solution of set (1) is available and that this solution satisfies the specified initial conditions. It is also assumed in this discussion that

$\delta x_i(t_o) = 0$, for all i; that is, that all initial conditions are specified. Let this solution be denoted by $\bar{x}_i = \bar{x}_i(t)$, $y_k = \bar{y}_k(t)$ and examine the behavior in the neighborhood of this solution by setting

$$x_i = \bar{x}_i + \delta x_i, \qquad i = 1, \ldots, n$$

$$y_k = \bar{y}_k + \delta y_k, \qquad k = 1, \ldots, m$$

and linearizing:

$$\delta\dot{x}_i = \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} \delta x_j + \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k} \delta y_k, \qquad i = 1, \ldots, n \tag{15}$$

The partial deriviatives of g_i are evaluated along the reference trajectory and are known functions of time. The functions δx_i and δy_k are the variations of x_i and y_k, respectively.

b. By definition, the following set of first-order differential equations is obtained from the homogeneous system of (15) by transposing the matrix of coefficients and changing the sign

$$\dot{\lambda}_j = - \sum_{i=1}^{n} \lambda_i \frac{\partial g_i}{\partial x_j}, \qquad j = 1, \ldots, n \tag{16}$$

This system is the adjoint system to (15). The solution of the two systems

$$\lambda_j, \qquad j = 1, \ldots, n$$

$$\delta x_j, \qquad j = 1, \ldots, n$$

are related by

$$\frac{d}{dt} \sum_{j=1}^{n} \lambda_j \, \delta x_j = \sum_{k=1}^{m} \sum_{j=1}^{n} \lambda_j \frac{\partial g_j}{\partial y_k} \delta y_k$$

This result may be verified by evaluating the derivatives to obtain

$$\frac{d}{dt} \sum_{j=1}^{n} \lambda_j \, \delta x_j = \sum_{j=1}^{n} \dot{\lambda}_j \, \delta x_j + \sum_{j=1}^{n} \lambda_j \, \delta \dot{x}_j \tag{17}$$

Substituting (15) and (16) into this expression

$$\begin{aligned} \frac{d}{dt} \sum_{j=1}^{n} \lambda_j \, \delta x_j = &- \sum_{j=1}^{n} \sum_{p=1}^{n} \lambda_p \frac{\partial g_p}{\partial x_j} \delta x_j \\ &+ \sum_{j=1}^{n} \sum_{p=1}^{n} \lambda_j \frac{\partial g_j}{\partial x_p} \delta x_p \\ &+ \sum_{j=1}^{n} \sum_{k=1}^{m} \lambda_j \frac{\partial g_j}{\partial y_k} \delta y_k \end{aligned} \tag{18}$$

Integrating (18) between definite limits t_o and t_f,

$$\begin{aligned} &\sum_{j=1}^{n} \lambda_j \left(t_f\right) \delta x_j \left(t_f\right) - \sum_{j=1}^{n} \lambda_j \left(t_o\right) \delta x_j \left(t_o\right) \\ &= \int_{t_o}^{t_f} \sum_{j=1}^{n} \sum_{k=1}^{m} \lambda_j \frac{\partial g_j}{\partial y_k} \delta y_k \, dt \end{aligned} \tag{19}$$

where

$$\sum_{j=1}^{n} \sum_{p=1}^{n} \lambda_p \frac{\partial g_p}{\partial x_j} \delta x_j = \sum_{j=1}^{n} \sum_{p=1}^{n} \lambda_j \frac{\partial g_j}{\partial x_p} \delta x_p$$

Consider the special solution corresponding to

$$\lambda_j \left(t_f\right) = 1, \qquad j = i$$

$$\lambda_j \left(t_f\right) = 0, \qquad j = i$$

and assign the symbols $\lambda_j^{(i)}(t)$. The value $\lambda_j^{(i)}(t)$ can be obtained by integrating (1) and (16) backward in time from t_f. Obtaining this result for all λ_j, n expressions are obtained for the $\delta x_i(t_f)$ values and from (19)

$$\delta x_i\left(t_f\right) = \sum_{j=1}^{n} \lambda_j^{(i)}\left(t_o\right) \delta x_j\left(t_o\right)$$

$$+ \int_{t_o}^{t_f} \sum_{k=1}^{m} \sum_{j=1}^{n} \lambda_j^{(i)} \frac{\partial g_j}{\partial y_k} \delta y_k \, dt, \qquad i = 1, \ldots, n$$

Since $\delta x_j\left(t_o\right)$ is defined to be equal to zero, the following expression is obtained for the change in $\delta x_i\left(t_f\right)$ from the nominal as a result of changes in the control variables:

$$\delta x_i\left(t_f\right) = \int_{t_o}^{t_f} \sum_{k=1}^{m} \sum_{j=1}^{n} \lambda_j^{(i)} \frac{\partial g_j}{\partial y_k} \delta y_k \, dt, \qquad i = 1, \ldots, m \tag{20}$$

The problem posed this far - that is, minimizing P as given by equation (3) subject to r constraints as given by equation (2) - can be approached in another manner. Replace the previous problem with the problem of minimizing

$$P' = P + \frac{1}{2} \sum_{j=1}^{r} k_j \psi_j^2 \tag{21}$$

The k_j are positive real numbers and are weighting factors on the squares of the errors between the ψ_j. This function is defined as the penalty function in that the constraints ψ_j will penalize P' when the last member of (21) is large.

6. SECOND VARIATION DEVELOPMENT

For this discussion let all initial values of the state variables and time be specified. The terminal time will be considered as free and let the terminal state values be subject to r constraints

$$\psi_j\left[x_i\left(t_f\right), \ldots, x_n\left(t_f\right)\right], \qquad j = 1, \ldots, r$$

Using the penalty functions, as defined in the preceding section, the end conditions at final time are considered free and the function to be minimized is

$$P' = P + \frac{1}{2} \sum_{j=1}^{r} K_j \psi_j^2$$

where P is defined by equation 3.

Consider a trajectory in space whose initial conditions satisfy the required values, and whose state variables are determined as functions of time by the set (1). The vector $y(t)$ is not optimal and is chosen as a set which will direct the trajectory in the desired direction. The trajectory is terminated at a relative minimum of P'; that is, when

$$\frac{dP'}{dt_f} = 0$$

the sign of $\left(\frac{dP'}{dt_f}\right)^-$ is negative, and the sign of $\left(\frac{dP'}{dt_f}\right)^+$ is positive The basis of choice of this terminal condition is twofold. First, since it is desired to minimize P', the trajectory will be terminated at a minimum of this function. The second reason will become apparent from the following discussion. The total derivative of P' with respect to time is given by the following expression:

$$\frac{dP'}{dt_f} = \frac{\partial P'}{\partial t_f} + \sum_{j=1}^{n} \left(\frac{\partial P'}{\partial x_{jf}}\right) \frac{dx_j}{dt_f}$$

$$= \frac{\partial P'}{\partial t_f} + \sum_{j=1}^{n} \left(\frac{\partial P'}{\partial x_{jf}}\right) g_{jf}$$

From the transversality condition (12),

$$\left(\frac{\partial P'}{\partial x_{if}}\right) = \lambda_{if}$$

Substituting this condition into the above equation,

$$\frac{dP'}{dt_f} = \frac{\partial P'}{\partial t_f} + \sum_{j=1}^{n} \lambda_{jf}\, g_{jf}$$

$$= \frac{\partial P'}{\partial t_f} + H_f$$

Since in rocket applications

$$\frac{dP'}{dt_f} = 0$$

it is seen from equation (11) that this condition also satisfies the transversality condition for final time.

It is clear that this trajectory not only does not meet the required final conditions, but is not optimal since not all of the conditions for a minimum are satisfied. In fact, the trajectory satisfies only system (1) and condition (11). It is desired to use this reference trajectory and calculate $\delta y(t)$ to obtain a new trajectory in the neighborhood of the reference trajectory for which a reduction in P' is realized.

If an expansion is made of P' along the reference trajectory, an estimate of the value of P' for the new trajectory can be obtained and is equal to

$$P' = P'_o + P_1' + \frac{1}{2} P_2' + \ldots$$

Here, P_o' is equal to the penalty function for the reference trajectory, P' is the penalty function for the new trajectory, P_1' is the collection of first-order terms in the variation of the state and control variables, and P_2' is the collection of second-order terms in the variations of the state and control variables. Higher order terms will be neglected for this discussion.

An expansion of P' in the neighborhood of the terminal point of the reference trajectory is given by

$$P' = P'\left(\bar{x}_{if}, \ldots, \bar{x}_{nf}, \bar{t}_f\right)$$

$$+ \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} \Delta x_{if} + \frac{\partial P'}{\partial t_f} \delta t_f$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 P'}{\partial x_{if}\, \partial x_{jf}} \Delta x_{if}\, \Delta x_{jf}$$

$$+ \sum_{i=1}^{n} \frac{\partial^2 P'}{\partial x_{if}\, \partial t_f} \Delta x_{if}\, \delta t_f + \frac{1}{2} \frac{\partial^2 P'}{\partial t_f^{\,2}} \delta t_f^{\,2} \tag{22}$$

where $\bar{x}_i$ denotes values on the reference trajectory.

It is desired to obtain an approximation to (22) valid to second-order in control variations $\delta y_k(t)$, and estimates of the terminal increments Δx_{if}, which are correct to second-order terms, are required. A first-order estimate of the variations is given by (15). Integration of this system to final time would yield a first-order estimate of δx_{if}. An equivalent estimate is given by (20). Since a second-order estimate is desired,

$$\delta \dot{x}_i = \sum_{p=1}^{n} \frac{\partial g_i}{\partial x_p} \delta x_p + \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k} \delta y_k + \frac{1}{2} \sum_{p=1}^{n} \sum_{g=1}^{n} \frac{\partial^2 g_i}{\partial x_p\, \partial x_g} \delta x_p\, \delta x_g$$

$$+ \sum_{p=1}^{n} \sum_{k=1}^{m} \frac{\partial^2 g_i}{\partial x_p\, \partial y_k} \delta x_p\, \delta y_k + \frac{1}{2} \sum_{k=1}^{m} \sum_{s=1}^{m} \frac{\partial^2 g_i}{\partial y_k\, \partial y_s} \delta y_k\, \delta y_s$$

Substituting this result into (17), the following expression is obtained:

$$\frac{d}{dt}\sum_{j=1}^{n}\lambda_j\,\delta x_j = \sum_{j=1}^{n}\sum_{p=1}^{n}\lambda_j\frac{\partial g_j}{\partial x_p}\delta x_p$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{m}\lambda_j\frac{\partial g_j}{\partial y_k}\delta y_k - \sum_{p=1}^{n}\sum_{j=1}^{n}\lambda_p\frac{\partial g_p}{\partial x_j}\delta x_j$$

$$+\frac{1}{2}\sum_{j=1}^{n}\sum_{p=1}^{n}\sum_{g=1}^{n}\lambda_j\frac{\partial^2 g_j}{\partial x_p\,\partial x_g}\delta x_p\,\delta x_g$$

$$+\sum_{j=1}^{n}\sum_{p=1}^{n}\sum_{k=1}^{m}\lambda_j\frac{\partial^2 g_j}{\partial x_p\,\partial y_k}\delta x_p\,\delta y_k$$

$$+\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{s=1}^{m}\lambda_j\frac{\partial^2 g_j}{\partial y_k\,\partial y_s}\delta y_k\,\delta y_s$$

Integrating this expression between limits t_o and t_f and setting $\delta x_i(t_o) = 0$ and noting that

$$\sum_{j=1}^{n}\sum_{p=1}^{n}\lambda_j\frac{\partial g_j}{\partial x_p}\delta x_p = \sum_{p=1}^{n}\sum_{j=1}^{n}\lambda_p\frac{\partial g_p}{\partial x_j}\delta x_j$$

the following expression is obtained:

$$\sum_{j=1}^{n} \lambda_j \left(t_f\right) \delta x_j \left(t_f\right) = \int_{t_o}^{t_f} \sum_{j=1}^{n} \sum_{k=1}^{m} \lambda_j \frac{\partial g_j}{\partial y_k} \delta y_k \, dt$$

$$+ \frac{1}{2} \int_{t_o}^{t_f} \sum_{j=1}^{n} \sum_{p=1}^{n} \sum_{g=1}^{n} \lambda_j \frac{\partial^2 g_j}{\partial x_p \, \partial x_g} \delta x_p \, \delta x_g \, dt$$

$$+ \int_{t_o}^{t_f} \sum_{j=1}^{n} \sum_{p=1}^{n} \sum_{k=1}^{m} \lambda_j \frac{\partial^2 g_j}{\partial x_p \, \partial y_k} \delta x_p \, \delta y_k \, dt$$

$$+ \frac{1}{2} \int_{t_o}^{t_f} \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{s=1}^{m} \lambda_j \frac{\partial^2 g_j}{\partial y_k \, \partial y_s} \delta y_k \, \delta y_s \, dt \qquad (23)$$

Again, as in section 4, use the special boundary condition

$$\lambda_j \left(t_f\right) = 1, \qquad j = i$$

$$\lambda_j \left(t_f\right) = 0, \qquad j = i$$

and using the sumbols $\lambda_j^{(i)}(t)$. The function

$$H_i = \sum_{j=1}^{n} \lambda_j^{(i)} g_j \qquad (24)$$

is defined in terms of the solution $\lambda_j^{(i)}$ of the adjoint system (16). With these definitions and the system (23), a second order estimate of the increments δx_i is given by the integral

$$\xi_i \left(t_f\right) = \int_{t_o}^{t_f} \left(\sum_{k=1}^{m} \frac{\partial H_i}{\partial y_k} \delta y_k + \omega_i \right) dt \qquad (25)$$

where

$$2\omega_i = \sum_{j=1}^{n} \sum_{p=1}^{n} \sum_{g=1}^{n} \lambda_j^{(i)} \frac{\partial^2 g_j}{\partial x_p \, \partial x_g} \delta x_p \, \delta x_g$$

$$+ 2 \sum_{j=1}^{n} \sum_{p=1}^{n} \sum_{k=1}^{m} \lambda_j^{(i)} \frac{\partial^2 g_j}{\partial x_p \, \partial y_k} \delta x_p \, \delta y_k$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{s=1}^{m} \lambda_j^{(i)} \frac{\partial^2 g_j}{\partial y_k \, \partial y_s} \delta y_k \, \delta y_s \qquad (26)$$

Expression (25) is a second order estimate of $\delta x_i(t_f)$. If final time is variable an estimate of $\Delta x_i(t_f)$ is given by

$$\Delta x_{if} = \xi_i(t_f) + g_{if} \, \delta t_f + \sum_{j=1}^{n} \frac{\partial g_{if}}{\partial x_{jf}} \delta x_{jf} \, \delta t_f$$

$$+ \sum_{k=1}^{m} \frac{\partial g_{if}}{\partial x_{jf}} \delta y_{kf} \, \delta t_f + \frac{1}{2} \sum_{j=1}^{n} \frac{\partial g_{if}}{\partial x_{if}} g_{jf} \, \delta t_f^{\,2}$$

$$+ \frac{1}{2} \sum_{k=1}^{m} \frac{\partial g_{if}}{\partial y_{kf}} \dot{y}_{kf} \, \delta t_f^{\,2} + \frac{1}{2} \frac{\partial g_{if}}{\partial t_f} \delta t_f^{\,2} \qquad (27)$$

Substituting equations (27) into (22) the following second-order approximation to P' is obtained:

$$J_o + J_1 + \frac{1}{2} J_2 = P'\left(\bar{x}_{if}, \ldots, \bar{x}_{nf}, \bar{t}_f\right) + \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} \Bigg[\varepsilon_{if}$$

$$+ \left(g_{if} + \sum_{j=1}^{n} \frac{\partial g_{if}}{\partial x_{jf}} \delta x_{jf} + \sum_{k=1}^{m} \frac{\partial g_{if}}{\partial y_{kf}} \delta y_f \right) \delta t_f$$

$$+ \frac{1}{2} \left(\sum_{j=1}^{n} \frac{\partial g_{if}}{\partial x_{jf}} g_{jf} + \sum_{k=1}^{m} \frac{\partial g_{if}}{\partial y_{kf}} \dot{y}_{kf} + \frac{\partial g_{if}}{\partial t_f} \right) \delta t_f^{\,2} \Bigg] + \frac{\partial P'}{\partial t_f} \delta t_f$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 P'}{\partial x_{if} \, \partial x_{jf}} \left(\delta x_{if} + g_{if} \, \delta t_f\right) \left(\delta x_{jf} + g_{jf} \, \delta t_f\right)$$

$$+ \sum_{i=1}^{n} \frac{\partial^2 P'}{\partial x_{if} \, \partial t_f} \left(\delta x_{if} + g_{if} \, \delta t_f\right) \delta t_f + \frac{1}{2} \frac{\partial^2 P'}{\partial t_f^{\,2}} \delta t_f^{\,2} \qquad (28)$$

The problem is to find δy_k values that will minimize the expression (28). The Hamiltonian H as defined in section 3 is related to the H_i terms by the following expression:

$$H = \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} H_i$$

where H_i is defined by (24).

Let ω be defined by the following expression:

$$\omega = \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} \omega_i$$

where ω_i is defined by (26).

Therefore,

$$\omega = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 H}{\partial x_i \, \partial x_j} \delta x_i \, \delta x_j$$

$$+ \sum_{i=1}^{n} \sum_{k=1}^{m} \frac{\partial^2 H}{\partial x_i \, \partial y_k} \delta x_i \, \delta y_k$$

$$+ \frac{1}{2} \sum_{k=1}^{m} \sum_{s=1}^{m} \frac{\partial^2 H}{\partial y_k \, \partial y_s} \delta y_k \, \delta y_s$$

The expression (28) which is to be minimized contains the expression

$$\sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} \xi_{if} = \int_{t_o}^{t_f} \left(\sum_{k=1}^{m} \frac{\partial H}{\partial y_k} \delta y_k + \omega \right) dt \tag{29}$$

where t_f is fixed.

Equation (28) is of the general form

$$\psi = G\left(t_f, x_{if}\right) + \int_{t_o}^{t_f} I\left(t, x_i, \dot{x}_i\right) dt \tag{30}$$

which corresponds to the problem of Bolza. The problem of Bolza is defined in the same manner as the problem of Mayer, except the function (3) to be minimized is exchanged for expression (30). This problem is a separate problem within the big problem. The differential constraits for this problem are given by (15). It is desired to adjoin these differential constraints with new multipliers $\delta\lambda_i$, $i = 1, \ldots, n$, and write the Euler-Lagrange equations, the Weierstrass necessary condition, and the transversality conditions for the problem of Bolza. The general Hamiltonian expression for the problem of Bolza is given by

$$h = I + \sum_{i=1}^{n} \text{(ith Lagrange multiplier) (ith variational equation).}$$

For this particular problem,

$$h = \sum_{i=1}^{n} \delta\lambda_i \left(\sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} \delta x_j + \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k} \delta y_k \right)$$

$$+ \sum_{k=1}^{m} \frac{\partial H}{\partial y_k} \delta y_k + \omega$$

The Weierstrass necessary condition takes the form

$$h\left(\delta y^*_1, \ldots, \delta y_m^*\right) \geq h\left(\delta y_1, \ldots, \delta y_m\right)$$

in which the δy_k^*, $k = 1, \ldots, m$, are arbitrary. This expression simply states that the δy_k, $k = 1, \ldots, m$, required to minimize (28) will yield a smaller or equal value of h than any other δy^*_k, $k = 1, \ldots, m$.

Corresponding to equation (13), the Euler-Lagrange conditions are given by

$$\delta\dot{\lambda}_i = - \frac{\partial h}{\partial \delta x_i}, \qquad i = 1, \ldots, n \tag{32}$$

Taking the indicated partials,

$$\delta\dot{\lambda}_i = - \sum_{j=1}^{n} \delta\lambda_j \frac{\partial g_j}{\partial x_i} - \frac{\partial \omega}{\partial \delta x_i}$$

$$= - \sum_{j=1}^{n} \delta\lambda_j \frac{\partial g_j}{\partial x_i} - \sum_{j=1}^{n} \frac{\partial^2 H}{\partial x_i \, \partial x_j} \delta x_j \tag{33}$$

$$- \sum_{k=1}^{m} \frac{\partial^2 H}{\partial x_i \, \partial y_k} \delta y_k, \qquad i = 1, \ldots, n$$

If equation (13) is linearized, the following expression is obtained:

$$\dot{\lambda}_i + \delta\dot{\lambda}_i = - \frac{\partial H}{\partial x_i} - \sum_{j=1}^{n} \frac{\partial^2 H}{\partial x_i \, \partial x_j} \delta x_j - \sum_{k=1}^{m} \frac{\partial^2 H}{\partial x_i \, \partial y_k} \delta y_k$$

$$- \sum_{j=1}^{n} \frac{\partial^2 H}{\partial x_i \, \partial \lambda_j} \delta\lambda_j, \qquad i = 1, \ldots, n$$

Since

$$\dot{\lambda}_i = - \frac{\partial H}{\partial x_i}$$

and

$$\frac{\partial^2 H}{\partial x_i \, \partial \lambda_j} = \frac{\partial g_j}{\partial x_i}$$

the above equation reduces to

$$\delta\dot{\lambda}_i = - \sum_{j=1}^{n} \frac{\partial g_j}{\partial x_i} \delta\lambda_j - \sum_{j=1}^{n} \frac{\partial^2 H}{\partial x_i \, \partial x_j} \delta x_j \tag{34}$$

$$- \sum_{k=1}^{m} \frac{\partial^2 H}{\partial x_i \, \partial y_k} \delta y_k, \qquad i = 1, \ldots, n$$

Comparing (33) and (34) it is seen that the linearized version of (13) is equal to the Euler equation for the Bolza problem. This, the $\delta\lambda_i$ multipliers are only linearized versions of the λ_i multipliers for the Mayer problem and the $\delta\dot{\lambda}_i$ equations can be obtained by calculating the linearized version of $\dot{\lambda}_i$.

The transversality conditions corresponding to open δx_{jf} are given by

$$\delta\lambda_{jf} = \frac{\partial\left(J_o + J_1 + \frac{1}{2} J_2\right)}{\partial \delta x_{jf}}, \quad j = 1, \ldots, n$$

Taking the appropriate derivative of (28) the following results are obtained:

$$\delta\lambda_{jf} = \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} \frac{\partial g_{if}}{\partial x_{jf}} \delta t_f + \sum_{i=1}^{n} \frac{\partial^2 P'}{\partial x_{if}\, \partial x_{jf}} \left(\delta x_{if} + g_{if}\, \delta t_f\right)$$

$$+ \frac{\partial^2 P'}{\partial x_{jf}\, \partial t_f} \delta t_f = 0, \qquad j = 1, \ldots, n \tag{35}$$

If a linearized version of (12) is calculated, the following result is obtained:

$$\lambda_{jf} + \Delta\lambda_{jf} = \lambda_{jf} + \delta\lambda_{jf} + \dot{\lambda}_j\, \delta t_f$$

$$= \frac{\partial P'}{\partial x_{jf}} + \sum_{i=1}^{n} \frac{\partial^2 P'}{\partial x_{jf}\, \partial x_{if}} \delta x_{if}$$

$$+ \frac{\partial^2 P'}{\partial x_{jf}\, \partial t_f} \delta t_f + \sum_{i=1}^{n} \frac{\partial^2 P'}{\partial x_{jf}\, \partial x_{if}} g_{if}\, \delta t_f$$

Since

$$\lambda_{jf} = \frac{\partial P'}{\partial x_{jf}}$$

and

$$\dot{\lambda}_{jf} = - \sum_{i=1}^{n} \lambda_{if} \frac{\partial g_{if}}{\partial x_j}$$

$$= - \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} \frac{\partial g_{if}}{\partial x_{jf}}$$

the following expression is obtained:

$$\delta\lambda_{jf} = \sum_{j=1}^{n} \frac{\partial^2 P'}{\partial x_{jf}\,\partial x_{if}} \left(\delta x_{if} + g_{if}\,\delta t_f\right) + \frac{\partial^2 P'}{\partial x_{jf}\,\partial t_f}\,\delta t_f$$

$$+ \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} \frac{\partial g_{if}}{\partial x_{jf}}\,\delta t_f, \qquad j = 1, \ldots, n \tag{36}$$

Comparing equation (35) and (36) it is seen that these two expressions are equal. Therefore, the transversality conditions for free δx_{jf} for the Bolza problem is a linearized version of the transversality condition for free x_{jf} for the Mayer problem.

Since t_f is fixed in the Bolza problem, no transversality condition is needed for final time. However, the parameter δt_f must be chosen which minimizes (28). This condition will be satisfied if δt_f is chosen such that

$$\frac{\partial\left(J_o + J_1 + \frac{1}{2} J_2\right)}{\partial \delta t_f} = 0$$

The above expression is equal to the following expression:

$$\sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} g_{if} + \frac{\partial P'}{\partial t_f} + \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} \left[\left(\sum_{j=1}^{n} \frac{\partial g_{if}}{\partial x_{jf}}\,\delta x_{jf}\right.\right.$$

$$\left. + \sum_{k=1}^{m} \frac{\partial g_{if}}{\partial y_{kf}}\,\delta y_{kf}\right) + \left(\sum_{j=1}^{n} \frac{\partial g_{if}}{\partial x_{jf}}\,g_{jf} + \sum_{k=1}^{m} \frac{\partial g_{if}}{\partial y_{kf}}\,\dot{y}_{kf}\right.$$

$$+ \frac{\partial g_{if}}{\partial t_f}\Big) \delta t_f + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 P'}{\partial x_{if}\, \partial x_{jf}} g_{if} \left(\delta x_{jf} + g_{jf}\, \delta t_f\right)$$

$$+ \sum_{i=1}^{n} \frac{\partial^2 P'}{\partial x_{if}\, \partial t_f} \left(\delta x_{if} + 2 g_{if}\, \delta t_f\right) + \frac{\partial^2 P'}{\partial t_f^{\,2}} \delta t_f = 0 \qquad (37)$$

A linearized version of the transversality condition (11), where

$$H_f = \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} g_{if}$$

for the Mayer problem is as follows:

$$\sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} g_{if} + \frac{\partial P'}{\partial t_f} + \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} \sum_{j=1}^{n} \frac{\partial g_{if}}{\partial x_{jf}} \delta x_{jf}$$

$$+ \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} \sum_{k=1}^{m} \frac{\partial g_{if}}{\partial y_{kf}} \delta y_{kf} + \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} \sum_{j=1}^{n} \frac{\partial g_{if}}{\partial x_{jf}} g_{jf}\, \delta t_f$$

$$+ \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} \sum_{k=1}^{m} \frac{\partial g_{if}}{\partial y_{kf}} \dot{y}_{kf}\, \delta t_f + \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} \frac{\partial g_{if}}{\partial t_f} \delta t_f$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 P'}{\partial x_{if}\, \partial x_{jf}} g_{if}\, \delta x_{jf} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 P'}{\partial x_{if}\, \partial x_{jf}} g_{if}\, g_{jf}\, \delta t_f$$

$$+ \sum_{i=1}^{n} \frac{\partial^2 P'}{\partial x_{if}\, \partial t_f} g_{if}\, \delta t_f + \sum_{i=1}^{n} \frac{\partial^2 P'}{\partial t_f\, \partial x_{if}} \delta x_{if}$$

$$+ \sum_{i=1}^{n} \frac{\partial^2 P'}{\partial t_f\, \partial x_{if}} g_{if}\, \delta t_f + \frac{\partial^2 P'}{\partial t_f^{\,2}} \delta t_f = 0$$

Rearranging terms,

$$\sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} g_{if} + \frac{\partial P'}{\partial t_f} + \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} \left[\left(\sum_{j=1}^{n} \frac{\partial g_{if}}{\partial x_{jf}} \delta x_{jf} + \sum_{k=1}^{m} \frac{\partial g_{if}}{\partial y_{kf}} \delta y_{kf} \right) \right.$$

$$\left. + \left(\sum_{j=1}^{n} \frac{\partial g_{if}}{\partial x_{jf}} g_{jf} + \sum_{k=1}^{m} \frac{\partial g_{if}}{\partial y_{kf}} \dot{y}_{kf} + \frac{\partial g_{if}}{\partial t_f} \right) \delta t_f \right]$$

$$+ \sum_{i=1}^{n} \frac{\partial^2 P'}{\partial x_{if}\, \partial t_f} \left(\delta x_{if} + 2 g_{if}\, \delta t_f \right) \tag{38}$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 P'}{\partial x_{if}\, \partial x_{jf}} g_{if} \left(\delta x_{jf} + g_{jf}\, \delta t_f \right) + \frac{\partial^2 P'}{\partial t_f^{\,2}} \delta t_f = 0$$

Comparing (37) and (38), it is seen that the condition for minimizing δt_f for the Bolza problem is the same as the linearized version of the transversality condition for free t_f for the Mayer problem.

Before continuing the discussion of conditions that must be satisfied to minimize (28), it is necessary to discuss the ideas presented in the next section.

7. RESTRICTION OF STEP SIZE OF CONTROL VARIABLES

The function h as given by (31) is a second-order quadratic approximation to H. If h is concave up a minimum will exist, but if h is concave down difficulties will occur in determining step size changes in the control variables since no minimum of h will exist. The following discussion will result in a modification to alleviate the difficulty.

Let the following integrals represent constraints on the step size changes in δy_k, $k = 1, \ldots, m$;

$$\int_{t_o}^{t_f} \frac{1}{2}\, \delta y_k^{\,2}\, dt = a_k^{\,2}, \qquad k = 1, \ldots, m.$$

An equivalent definition of the above integrals are the following forms:

$$\delta\dot{x}_{(n + k)} = \frac{1}{2}\, \delta y_k^{\,2}, \qquad k = 1, \ldots, m$$

where

$$\delta x_{(n + k)}\left(t_o\right) = 0$$

$$\delta x_{(n + K)}\left(t_f\right) = a_k^{\,2}$$

The above $\delta x_{n + k}$ values represent additional state equations and the problem must be completely reformulated with the addition of the m state variables. However, the Euler-Lagrange equations and the transversality conditions previously developed are identical to those for the modified problem. The function h now becomes

$$h = \sum_{i = 1}^{n} \delta\lambda_i \left(\sum_{j = 1}^{n} \frac{\partial g_i}{\partial x_j}\, \delta x_j + \sum_{k = 1}^{m} \frac{\partial g_i}{\partial y_k}\, \delta y_k \right)$$

$$+ \sum_{k = 1}^{m} \frac{\partial H}{\partial y_k}\, \delta y_k + \omega + \frac{1}{2} \sum_{k = 1}^{m} \delta\lambda_{n + k}\, \delta y_k^{\,2} \qquad (39)$$

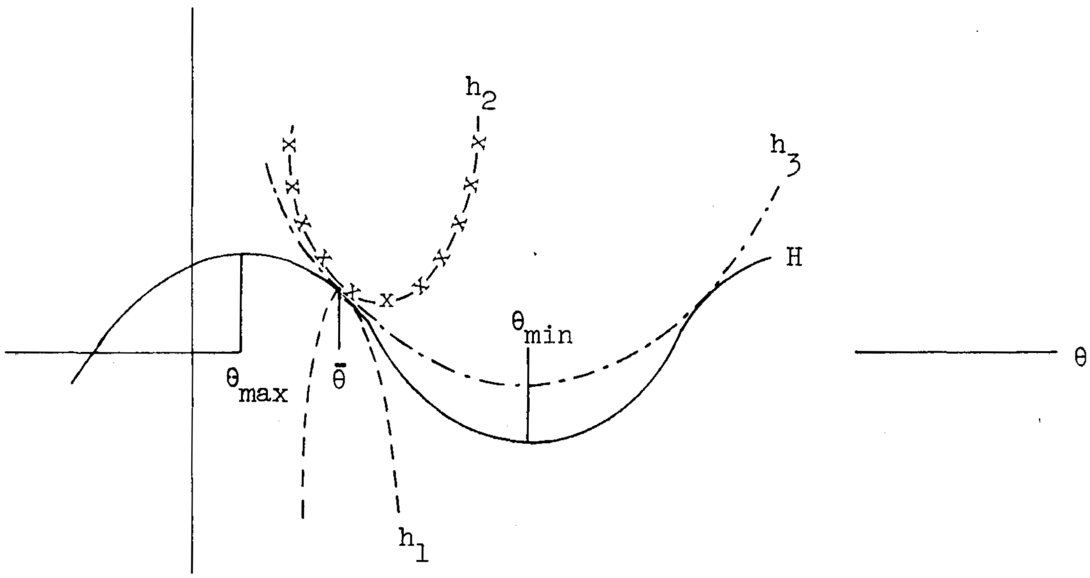
Since h is a quadratic equation the requirement for h to possess a minimum for unbounded control variables is

$$\frac{\partial^2 h}{\partial \delta y_k^{\,2}} = \frac{\partial^2 H}{\partial y_k^{\,2}} + \delta\lambda_{n + k} \geq 0 \qquad (40)$$

If $\dfrac{\partial^2 H}{\partial y_k^{\,2}}$ is negative a $\delta\lambda_{n + k}$ must be chosen at least large enough to cause (40) to be positive.

From computational experience, it has been determined that the above requirement, although correct, is not by itself adequate. That is, not only should h possess a minimum but also some method of controlling where the minimum should fall must be included. The following discussion will yield the needed results.

In applications to trajectory design, the Hamiltonian H plotted against a control θ will have the shape of a sine curve as shown in the figure below. The value of θ_{min} would be the desired



Quadratic Approximations of H

value to minimize H. This value can be determined by calculating the two values that satisfy the condition $\frac{\partial H}{\partial \theta} = 0$, that is, θ_{min} and θ_{max}. A comparison will yield the smallest value of H. The value $\bar{\theta}$ represents the current value of the control function. The curve h_1 in the above figure represents h if no constraints have been introduced to ensure a minimum. The curve h_2 is typical of the h function if condition (40) is employed to ensure a minimum. The desired h characteristic is a function such as h_3 which has the same minimum as H. It is possible to calculate a $\delta\lambda_\theta$ which will insure coincidence of the minima. Knowing the value of H at $\bar{\theta}$ the value of $H(\theta_{min})$ is approximated by the following second-order Taylor series expansion:

$$H(\theta_{min}) = H(\bar{\theta}) + \frac{\partial H}{\partial \theta}\,\delta\theta + \frac{1}{2}\,\frac{\partial^2 H}{\partial \theta^2}\,\delta\theta^2$$

$$= H(\bar{\theta}) + \frac{\partial H}{\partial \theta}\left(\theta_{min} - \bar{\theta}\right) + \frac{1}{2}\,\frac{\partial^2 H}{\partial \theta^2}\left(\theta_{min} - \bar{\theta}\right)^2$$

where the partial derivatives are evaluated at $\bar{\theta}$.

The above expression can be written as

$$\delta H = \frac{\partial H}{\partial \theta}\left(\theta_{min} - \bar{\theta}\right) + \frac{1}{2}\,\frac{\partial^2 H}{\partial \theta^2}\left(\theta_{min} - \bar{\theta}\right)^2$$

where

$$\delta\theta = \theta_{min} - \bar{\theta}$$

The above expression is only a quadratic equation in $\delta\theta$. To insure the expression has a minimum the coefficient of the $\delta\theta^2$ terms must be positive. This is insured by the previous argument if the equation is written as

$$\delta H = \frac{\partial H}{\partial \theta}\left(\theta_{min} - \bar{\theta}\right) + \frac{1}{2}\left(\frac{\partial^2 H}{\partial \theta^2} + \delta\lambda_\theta\right)\left(\theta_{min} - \bar{\theta}\right)^2$$

where $\delta\lambda_\theta$ is chosen to insure $\left(\frac{\partial^2 H}{\partial \theta^2} + \delta\lambda_\theta\right) > 0.$

If this quadratic is to have a minimum at θ_{min},

$$0 = \frac{\partial \delta H}{\partial \delta\theta} = \frac{\partial H}{\partial \theta} + \left(\frac{\partial^2 H}{\partial \theta^2} + \delta\lambda_\theta\right)\left(\theta_{min} - \bar{\theta}\right)$$

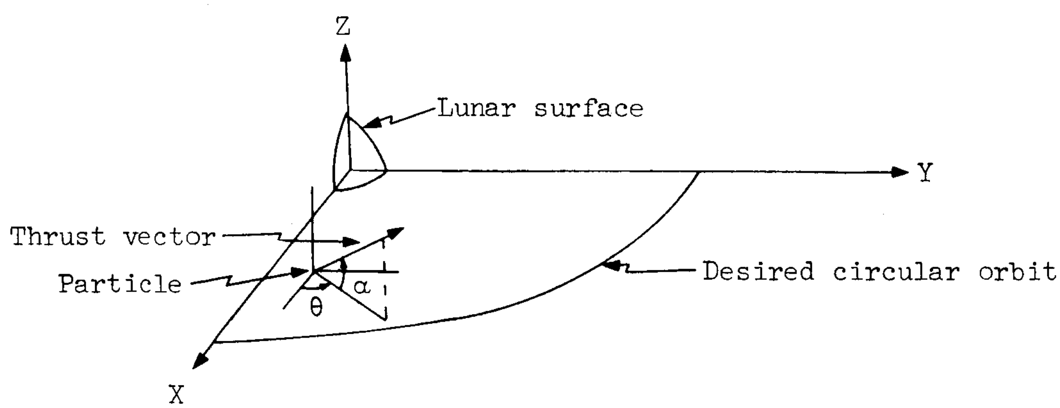
and the required value of $\delta\lambda_\theta$ for the minimum of h to fall on the same θ as the minimum of H is

$$\delta\lambda_\theta = \frac{-\frac{\partial H}{\partial \theta}}{\theta_{min} - \bar{\theta}} - \frac{\partial^2 H}{\partial \theta^2}$$

A similar result must be derived for each unbounded control function.

8. GEOMETRY AND EQUATIONS OF MOTION FOR INSERTION INTO CIRCULAR ORBIT

To help clarify the ideas presented in this memorandum, the derived equations are applied to a particular example. The equations of motion are needed before the ideas in the next section can be discussed. Insertion into a desired circular orbit to minimize fuel consumption with variable thrust capability is typical of most Apollo trajectories and only relatively minor changes are needed to solve other problems. Let the x, y plane be defined in the plane of the orbit. The angle θ defines the thrusting angle in a plane through the particle and parallel to the x, y plane. The angle θ is



Coordinate System

measured from a line parallel to the x axis. The angle α is the out-of-plane angle. The equations of motion are in the inertial system. Since the problem of Mayer calls for first-order differential equations, the equations of motion are of the following form:

$$g_1 = \dot{u} = \frac{T}{m} \cos \alpha \cos \theta - \mu x/R^3 \tag{41}$$

$$g_2 = \dot{v} = \frac{T}{m} \cos \alpha \sin \theta - \mu y/R^3 \tag{42}$$

$$g_3 = \dot{w} = \frac{T}{m} \sin \alpha - \mu z/R^3 \tag{43}$$

$$g_4 = \dot{x} = u \tag{44}$$

$$g_5 = \dot{y} = v \tag{45}$$

$$g_6 = \dot{z} = w \tag{46}$$

and

$$g_7 = \dot{m} = \frac{-T}{c} \tag{47}$$

where

T = thrust

m = mass of particle

μ = gravitation constant

$R^2 = x^2 + y^2 + z^2$

and

c = velocity of exhaust gas

The Hamiltonian which is defined as

$$H = \sum_{i=1}^{n} \lambda_i g_i$$

is equal to

$$\left.\begin{aligned} H = {} & \lambda_u \left(\frac{T}{m} \cos \alpha \cos \theta - \mu x/R^3 \right) \\ & + \lambda_v \left(\frac{T}{m} \cos \alpha \sin \theta - \mu y/R^3 \right) \\ & + \lambda_w \left(\frac{T}{m} \sin \alpha - \mu z/R^3 \right) \\ & + \lambda_x u + \lambda_y v + \lambda_z w + \lambda_m \left(\frac{-T}{c} \right) \end{aligned}\right\} \tag{48}$$

9. LEGENDRE-CLEBSCH NECESSARY CONDITIONS FOR A MINIMUM

An additional condition that must be satisfied for a minimum with one control variable θ is the Legendre-Clebsch condition

$$\frac{\partial^2 h}{\partial \delta \theta^2} \geq 0$$

From (40), since

$$\frac{\partial^2 h}{\partial \delta \theta^2} = \frac{\partial^2 H}{\partial \theta^2} + \delta \lambda_\theta \geq 0$$

this condition is satisfied. For two control variables, θ and α, the Legendre - Clebsch condition is

$$\frac{\partial^2 h}{\partial \delta \theta^2} \delta \theta^4 + 2 \frac{\partial^2 h}{\partial \delta \theta \partial \delta \alpha} \delta \theta^2 \delta \alpha^2 + \frac{\partial^2 h}{\partial \delta \alpha^2} \delta \alpha^4 \geq 0$$

Positive semidefiniteness of this quadratic form requires that

$$\left.\begin{aligned} &\frac{\partial^2 h}{\partial \delta \theta^2} \geq 0 \\ &\frac{\partial^2 h}{\partial \delta \alpha^2} \geq 0 \\ &\left(\frac{\partial^2 h}{\partial \delta \theta^2}\right)\left(\frac{\partial^2 h}{\partial \delta \alpha^2}\right) - \left(\frac{\partial^2 h}{\partial \delta \theta \partial \delta \alpha}\right)^2 \geq 0 \end{aligned}\right\} \qquad (49)$$

From (40) and the condition that

$$\frac{\partial^2 H}{\partial \theta \partial \alpha} = \frac{\partial^2 h}{\partial \delta \theta \partial \delta \alpha}$$

the conditions can be written as

$$\frac{\partial^2 H}{\partial \theta^2} + \delta\lambda_\theta \geq 0 \tag{50}$$

$$\frac{\partial^2 H}{\partial \alpha^2} + \delta\lambda_\alpha \geq 0 \tag{51}$$

$$\left(\frac{\partial^2 H}{\partial \theta^2} + \delta\lambda_\theta\right)\left(\frac{\partial^2 H}{\partial \alpha^2} + \delta\lambda_\alpha\right) - \left(\frac{\partial^2 H}{\partial \theta \partial \alpha}\right)^2 \geq 0 \tag{52}$$

Since (40) assures satisfying (50) and (51), only condition (52) must be checked. If (52) is not satisfied, $\delta\lambda_\theta$ and $\delta\lambda_\alpha$ must be made large enough to satisfy the condition.

10. SWITCHING FUNCTIONS FOR SECOND VARIATIONAL METHODS

If the thrust level can be varied between minimum and maximum levels, it is desired to formulate switching functions corresponding to the discussion in section 4. If the time t_1 is determined as an optimum time to change the thrust level to minimize P', the following relationship must hold:

$$\frac{dP'}{dt_1} = 0$$

This equation is equal to

$$\frac{dP'}{dt_1} = \frac{\partial P'}{\partial t_1} + \sum_{i=1}^{n} \frac{\partial P'}{\partial x_i} \frac{dx_i}{dt_1}$$

Before proceeding, it will be necessary to evaluate $\frac{\partial P'}{\partial x_i}$ and $\frac{dx_i}{dt_1}$. The following derivation will result in an evaluation of the $\frac{\partial P'}{\partial x_i}$ terms.

If the nominal trajectory is optimal, the following relationship involving the variation δx_i and the multipliers $\lambda_i(t_1)$ holds:

$$\sum_{i=1}^{n} \lambda_i \, \delta x_i = \text{constant} \tag{53}$$

This may be verified by first taking the derivative of the left hand side and substituting the relations

$$\dot{\lambda}_i = \frac{-\partial H}{\partial x_i}$$

and

$$\delta \dot{x}_i = \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} \delta x_j + \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k} \delta y_k$$

That is,

$$\frac{d}{dt} \sum_{i=1}^{n} \lambda_i \, \delta x_i = \sum_{i=1}^{n} \dot{\lambda}_i \, \delta x_i + \sum_{i=1}^{n} \lambda_i \, \delta \dot{x}_i$$

$$= \sum_{i=1}^{n} - \frac{\partial H}{\partial x_i} \delta x_i + \sum_{i=1}^{n} \lambda_i \left(\sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} \delta x_j + \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k} \delta y_k \right)$$

$$= \sum_{i=1}^{n} - \frac{\partial H}{\partial x_i} \delta x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \frac{\partial g_i}{\partial x_j} \delta x_j + \sum_{i=1}^{n} \sum_{k=1}^{m} \lambda_i \frac{\partial g_i}{\partial y_k} \delta y_k$$

Since

$$\frac{\partial H}{\partial x_j} = \sum_{i=1}^{n} \lambda_i \frac{\partial g_i}{\partial x_j}$$

and

$$\frac{\partial H}{\partial y_k} = \sum_{i=1}^{n} \lambda_i \frac{\partial g_i}{\partial y_k}$$

the above expression can be reduced to

$$\frac{d}{dt} \sum_{i=1}^{n} \lambda_i \, \delta x_i = \sum_{i=1}^{n} - \frac{\partial H}{\partial x_i} \delta x_i + \sum_{j=1}^{n} \frac{\partial H}{\partial x_j} \delta x_j + \sum_{k=1}^{m} \frac{\partial H}{\partial y_k} \delta y_k$$

$$= \sum_{k=1}^{m} \frac{\partial H}{\partial y_k} \delta y_k$$

Since the nominal was assumed to be optimal,

$$\frac{\partial H}{\partial y_k} = 0, \qquad k = 1, \ldots, m$$

and

$$\frac{d}{dt} \sum_{i=1}^{m} \lambda_i \, \delta x_i = 0$$

which proves (53).

In particular,

$$\sum_{i=1}^{m} \lambda_i (t) \, \delta x_i (t) = \sum_{i=1}^{n} \lambda_{if} \, \delta x_{if}$$

Since

$$\Delta x_{if} = \delta x_{if} + g_{if} \, \delta t_f$$

and

$$H = \sum_{i=1}^{n} \lambda_i \, g_i$$

the above expression can be written as

$$\sum_{i=1}^{n} \lambda_i (t)\, \delta x_i (t) = \sum_{i=1}^{n} \lambda_{if} \left(\Delta x_{if} - g_{if}\right)$$

$$= \sum_{i=1}^{n} \lambda_{if}\, \Delta x_{if} - \sum_{i=1}^{n} \lambda_{if}\, g_{if}\, \delta t_f$$

$$= \sum_{i=1}^{n} \lambda_{if}\, \Delta x_{if} - H\left(t_f\right) \delta t_f$$

From section 3, for an optimum trajectory

$$H + \partial P' / \partial t_f = 0$$

and

$$\partial P' / \partial x_{if} = \lambda_{if}$$

so that

$$\sum_{i=1}^{n} \lambda_i (t)\, \delta x_i (t) = \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{if}} \Delta x_{if} + \frac{\partial P'}{\partial t_f} \delta t_f$$

The right-hand side of this equation is an expression for the change in P' which implies that

$$\lambda_i = \frac{\partial P'}{\partial x_i}$$

at any time t. Therefore the expression for $\frac{dP'}{dt_1}$ is reduced to

$$\frac{dP'}{dt_1} = \sum_{i=1}^{n} \lambda_i \left(t_1\right) \frac{dx_i\left(t_1\right)}{dt_1} \qquad (54)$$

where

$$\frac{\partial P'}{\partial t_1} = 0$$

The term $\frac{dx_i(t_1)}{dt_1}$ can be approximated from the equations of motion. From equation (41), if the switching time t_1 is changed by a small incremental value δt_1, the change in u is approximated by

$$\delta u = \left(\frac{T}{m} \cos \alpha \cos \theta - \frac{\mu x}{R^3}\right)^- \delta t_1$$

$$= \left(\frac{T^-}{m} \cos \theta \cos \alpha - \frac{\mu x}{R^3}\right) \delta t_1$$

However, the change in u at $t_1 + \delta t_1$ with the thrust level charged at t_1 would be

$$\delta u = \left(\frac{T}{m} \cos \theta \cos \alpha - \frac{\mu x}{R^3}\right)^+ \delta t_1$$

$$= \left(\frac{T^+}{m} \cos \theta \cos \alpha - \frac{\mu x}{R^3}\right) \delta t_1$$

The effective change in u would be

$$\delta u = \left(\frac{T^+ - T^-}{m} \cos \theta \cos \alpha\right) \delta t_1$$

or

$$\frac{du}{dt_1} = \frac{\left(T^+ - T^-\right)}{m} \cos \theta \cos \alpha$$

An equivalent result for the other state variables are the following partials:

$$\frac{dv}{dt_1} = \left(\frac{T^+ - T^-}{m}\right) \sin \theta \cos \alpha$$

$$\frac{dw}{dt_1} = \left(\frac{T^+ - T^-}{m}\right) \sin \alpha$$

$$\frac{dm}{dt_1} = -\left(\frac{T^+ - T^-}{c}\right)$$

$$\frac{dx}{dt_1} = 0$$

$$\frac{dy}{dt_1} = 0$$

$$\frac{dz}{dt_1} = 0$$

Equation (54) now takes the following form:

$$\frac{dP'}{dt_1} = \lambda_u \left[\frac{\left(T^+ - T^-\right)}{m}\right] \cos\theta \cos\alpha$$

$$+ \lambda_v \left[\frac{\left(T^+ - T^-\right)}{m}\right] \sin\theta \cos\alpha$$

$$+ \lambda_w \left[\frac{\left(T^+ - T^-\right)}{m}\right] \sin\alpha$$

$$+ \lambda_m \left[- \frac{\left(T^+ - T^-\right)}{c}\right]$$

A linearized version of the equation takes the following form:

$$\frac{dP'}{dt_1} + \sum_{j=1}^{n} \frac{\partial^2 P'}{\partial t_1 \, \partial \lambda_j} \Delta\lambda_j + \sum_{k=1}^{m} \frac{\partial^2 P'}{\partial t_1 \, \partial y_k} \Delta y_k = 0$$

Taking the appropriate derivatives, the above function takes the following forms

$$\begin{aligned} &\lambda_u \cos\theta \cos\alpha + \lambda_v \sin\theta \cos\alpha - \frac{1}{m}\Big(\lambda_u \cos\theta \cos\alpha + \lambda_v \sin\theta \cos\alpha \\ &+ \lambda_w \sin\alpha\Big) \Delta m + \lambda_w \sin\alpha - \frac{m\lambda_m}{c} + \cos\theta \cos\alpha \, \Delta\lambda_u + \sin\alpha \, \Delta\lambda_u \\ &+ \sin\theta \cos\alpha \, \Delta\lambda_v - \frac{m\Delta\lambda_m}{c} - \Big(\lambda_u \cos\theta \sin\alpha + \lambda_v \sin\alpha \sin\theta \\ &- \lambda_w \cos\alpha\Big)\Delta\alpha - \Big(\lambda_u \sin\theta \cos\alpha - \lambda_v \cos\alpha \cos\theta\Big)\Delta\theta = 0 \end{aligned} \tag{55}$$

From section 4, the condition for changing the thrust level is $\frac{\partial H}{\partial T} = 0$. For the geometry assumed in this memorandum this function is as follows:

$$\frac{\partial H}{\partial T} = \frac{\lambda_u}{m} \cos \alpha \cos \theta + \frac{\lambda_v}{m} \cos \alpha \sin \theta + \frac{\lambda_w}{m} \sin \alpha - \frac{\lambda_m}{c}$$

A linearized version of this expression is as follows:

$$\begin{aligned}
&\lambda_u \cos \alpha \cos \theta + \lambda_v \cos \alpha \sin \theta + \lambda_w \sin \alpha \\
&- \frac{m\lambda_m}{c} + \cos \alpha \cos \theta \, \Delta\lambda_u + \cos \alpha \sin \theta \, \Delta\lambda_v \\
&+ \sin \alpha \, \Delta\lambda_w - \frac{m\Delta\lambda_m}{c} + \left(- \lambda_u \cos \alpha \sin \theta + \lambda_v \cos \alpha \cos \theta\right) \Delta\theta \qquad (56) \\
&+ \left(- \lambda_u \sin \alpha \cos \theta - \lambda_v \sin \alpha \sin \theta + \lambda_w \cos \alpha\right) \Delta\alpha \\
&- \frac{1}{m} \left(\lambda_u \cos \alpha \cos \theta + \lambda_v \cos \alpha \sin \theta + \lambda_w \sin \alpha\right) \Delta m = 0
\end{aligned}$$

Comparing (55) and (56), it is seen that the switching functions are linearized versions of the switching functions for the Mayer problem.

11. EULER EQUATIONS FOR CONTROL VARIABLES

For an optimum trajectory, the following expressions are satisfied:

$$\frac{\partial h}{\partial \delta y_k} = 0, \qquad k = 1, \ldots, m$$

From (39)

$$\begin{aligned}
\frac{\partial h}{\partial \delta y_k} = &\sum_{i=1}^{n} \delta\lambda_i \frac{\partial g_i}{\partial y_k} + \frac{\partial H}{\partial y_k} \\
&+ \sum_{i=1}^{n} \frac{\partial^2 H}{\partial x_i \partial y_k} \delta x_i + \sum_{s=1}^{m} \frac{\partial^2 H}{\partial y_k \partial y_s} \delta y_s \qquad (57) \\
&+ \delta\lambda_{n+k} \delta y_k, \qquad k = 1, \ldots, m
\end{aligned}$$

For the Mayer problem, the Euler equation for the control variables are as follows:

$$\frac{\partial H}{\partial y_k} = 0, \qquad k = 1, \ldots, m$$

or

$$\frac{\partial H}{\partial y_k} = \sum_{i=1}^{n} \lambda_i \frac{\partial g_i}{\partial y_k}, \qquad k = 1, \ldots, m$$

Linearizing this expression

$$\begin{aligned} \frac{\partial H}{\partial y_k} &+ \sum_{i=1}^{n} \frac{\partial^2 H}{\partial y_k \partial x_i} \delta x_i + \sum_{s=1}^{m} \frac{\partial^2 H}{\partial y_k \partial y_s} \delta y_s \\ &+ \sum_{i=1}^{n} \frac{\partial^2 H}{\partial y_k \partial \lambda_i} \delta \lambda_i = 0 \end{aligned} \tag{58}$$

Since

$$\sum_{i=1}^{n} \frac{\partial^2 H}{\partial y_k \partial \lambda_i} \delta \lambda_i = \sum_{i=1}^{n} \frac{\partial g_i}{\partial y_k} \delta \lambda_i$$

equation (57) and (58) differ only by the $\delta\lambda_{n+k}$ term. Therefore, the Euler equation for the control variables for the Bolza problem are linearized versions of the Euler equation for the control variables with the addition of the $\delta\lambda_{n+k}$ terms.

12. RESULTING EQUATIONS FOR SECOND VARIATION AS APPLIED TO ORBIT INSERTION PROBLEMS

For the geometry and problem defined in section 8, the following five end conditions are required for insertion into circular orbit:

$$\psi_1 = u_f^2 + v_f^2 - \mu/\bar{R} = 0$$

$$\psi_2 = u_f x_f + v_f y_f = 0$$

$$\psi_3 = w_f = 0$$

$$\psi_4 = \sqrt{\left(x_f^2 + y_f^2\right)} - \bar{R} = 0$$

$$\psi_5 = z_f = 0$$

where $\bar{R}$ is the radius of the desired circular orbit, $\sqrt{\mu/\bar{R}}$ is the desired circular velocity.

The condition ψ_2 is the vector dot product and requires the final velocity vector and position vector to be perpendicular.

For this problem it is desired to maximize the final weight. Since the formulation has been for a minimum problem, the problem is changed to minimize the negative of the final weight. For computational purposes it was found better to minimize the following expression:

$$P = - \frac{m_f}{m_o}$$

where

m_o is the initial mass

and

m_f is the final mass.

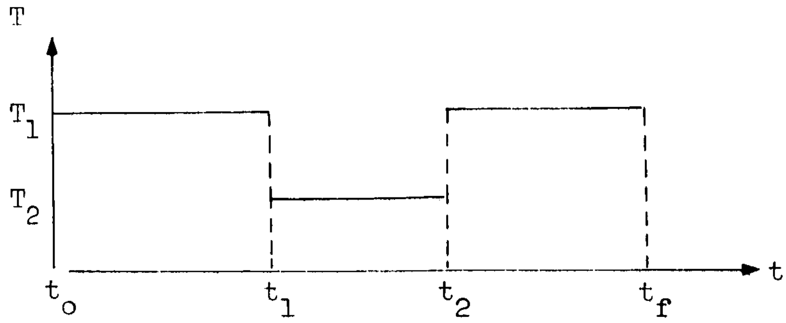
From the definition of the penalty function in section 5 and the above defined end conditions,

$$P' = \frac{-m_f}{m_o} + \frac{k_1}{2}\left(u_f^2 + v_f^2 - \frac{\mu}{\bar{R}}\right)^2$$

$$+ \frac{k_2}{2}\left(u_f\, x_f + v_f\, y_f\right)^2 + \frac{k_3}{2}\left(w_f^2\right)$$

$$+ \frac{k_4}{2}\left[\left(x_f^2 + y_f^2\right)^{1/2} - \bar{R}\right]^2 + \frac{k_5}{2}\, z_f^2$$

The thrusting scheme is assumed to be as follows:



Thrust Versus Time

where

$$T_1 = \text{maximum thrust}$$

$$T_2 = \text{minimum thrust}$$

The problem is to determine values of $\alpha(t)$, $\theta(t)$, t_1, and t_2 to minimize P' subject to the equations of motion.

The equations for the above Mayer problem are first developed since they are needed both to develop and solve the second variational equations. The Euler equations for the above problem are given by equation (13) and are of the form

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i}, \qquad i = 1, \ldots, n$$

and H is given by equation (48).

Taking the indicated partials, the following Euler equations are obtained:

$$\dot{\lambda}_u = -\lambda_x \tag{59}$$

$$\dot{\lambda}_v = -\lambda_y \tag{60}$$

$$\dot{\lambda}_w = -\lambda_z \tag{61}$$

$$\dot{\lambda}_x = \frac{-\mu}{R^5}\left[\lambda_u\left(3x^2 - R^2\right) + 3\lambda_v\, xy + 3\lambda_w\, xz\right] \tag{62}$$

$$\dot{\lambda}_y = \frac{-\mu}{R^5}\left[3\lambda_u\, xy + \lambda_v\left(3y^2 - R^2\right) + 3\lambda_w\, yz\right] \tag{63}$$

$$\dot{\lambda}_z = \frac{-\mu}{R^5}\left[3\lambda_u\, xz + 3\lambda_v\, yz + \lambda_w\left(3z^2 - R^2\right)\right] \tag{64}$$

and

$$\dot{\lambda}_m = \frac{T}{m^2}\left(\lambda_u \cos\alpha\cos\theta + \lambda_v \cos\alpha\sin\theta + \lambda_w \sin\alpha\right) \tag{65}$$

From equation (12), the transversality condition for free x_{i_f} is of the following form:

$$\frac{\partial P'}{\partial x_{i_f}} = \lambda_{i_f}$$

Taking the indicated partials, the following transversality conditions are obtained:

$$\lambda_u\left(t_f\right) = k_1\, u_f\left[1 - \left(\frac{\mu}{\overline{R}}\right)^{1/2}\left(u_f^{\,2} + v_f^{\,2}\right)^{-1/2}\right] + k_2\, x_f\left(u_f\, x_f + v_f\, y_f\right) \tag{66}$$

$$\lambda_v\left(t_f\right) = k_1\, v_1\left\{1 - \left[\frac{\mu}{\overline{R}\left(u_f^{\,2} + v_f^{\,2}\right)}\right]^{1/2}\right\} + k_2\, y_f\left(u_f\, x_f + v_f\, y_f\right) \tag{67}$$

$$\lambda_w\left(t_f\right) = k_3\, w_f \tag{68}$$

$$\lambda_x\left(t_f\right) = k_2\, u_f\left(u_f\, x_f + f_f\, v_f\, y_f\right) + k_4\, x_f\left[1 - \bar{R}\left(x_f^{\,2} + y_f^{\,2}\right)^{-1/2}\right] \tag{69}$$

$$\lambda_y\left(t_f\right) = k_2\, v_f\left(u_f\, x_f + v_f\, y_f\right) + k_4 y_f\left[1 - \bar{R}\left(x_f^{\,2} + y_f^{\,2}\right)^{-1/2}\right] \tag{70}$$

$$\lambda_z\left(t_f\right) = k_5\, z_f \tag{71}$$

and

$$\lambda_m\left(t_f\right) = -\frac{1}{m_o} \tag{72}$$

The Euler equations for the control variables are given by equation (14) and have the following form:

$$\frac{\partial H}{\partial \theta} = \frac{T}{m}\left(-\lambda_u \cos\alpha \sin\theta + \lambda_v \cos\alpha \cos\theta\right) \tag{73}$$

and

$$\frac{\partial H}{\partial \alpha} = \frac{T}{m}\left(-\lambda_u \sin\alpha \cos\theta - \lambda_v \sin\alpha \sin\theta + \lambda_w \cos\alpha\right) \tag{74}$$

The transversality condition for free final time is given by equation (11), but since $\frac{\partial P'}{\partial t_f} = 0$, the condition reduces to

$$H_f = 0 \tag{75}$$

To apply the second variational results, P' should be expanded into the form (28). However, from the previous results, all of the second variational equations are linearized versions of the equations developed for the Mayer problem. The differential equations for the state variables are obtained by linearizing the differential equations (41) through (47), and the general form is given by equation (15). Performing the indicated operations, the following differential equations are obtained:

$$\delta\dot{u} = \frac{\mu}{R^5}\left[\left(3x^2 - R^2\right)\delta x + 3xy\,\delta y + 3xz\,\delta z\right]$$

$$- \frac{T}{m}\left(\sin\alpha\cos\theta\,\delta\alpha + \cos\alpha\sin\theta\,\delta\theta\right)$$

$$- \frac{T}{m^2}\cos\alpha\cos\theta\,\delta m \tag{76}$$

$$\delta\dot{v} = \frac{u}{R^5}\left[3yx\,\delta x + \left(3y^2 - R^2\right)\delta y + 3y\,z\delta z\right]$$

$$- \frac{T}{m}\left(\sin\alpha\sin\theta\,\delta\alpha + \cos\alpha\cos\theta\,\delta\theta\right)$$

$$- \frac{T}{m^2}\cos\alpha\sin\theta\,\delta m \tag{77}$$

$$\delta\dot{w} = \frac{\mu}{R^5}\left[3z\,x\delta x + 3zy\,\delta y + \left(3z^2 - R^2\right)\delta z\right]$$

$$+ \frac{T}{m}\cos\alpha\,\delta\alpha - \frac{T}{m^2}\sin\alpha\delta m \tag{78}$$

$$\delta\dot{x} = \delta u \tag{79}$$

$$\delta\dot{y} = \delta v \tag{80}$$

$$\delta\dot{z} = \delta w \tag{81}$$

and

$$\delta\dot{m} = 0 \tag{82}$$

The term δm must be treated separately since $\delta\dot{m} = 0$ and δm varies only if δt_1 or δt_2 varies. The expression for δm is given by the following:

$$\delta m = -\hat{H}\left(t_1\right)\left(\frac{T_1 - T_2}{c}\right)\delta t_1$$

$$-\hat{H}\left(t_2\right)\left(\frac{T_2 - T_1}{c}\right)\delta t_2$$

where $\hat{H}$ is the Heaviside function and is defined as

$$\hat{H}\left(t_i\right) = \begin{matrix} 0 \\ 1 \end{matrix}\Bigg] \quad \begin{matrix} t < t_i \\ t_i \leq t \end{matrix} \qquad i = 1, 2 \tag{83}$$

From the previous discussion, the second variational Euler equations are linearized versions of the Euler equations for the Mayer problems, and the general expression is given by (34). Taking linearized versions of equations (59) through (65), the following second variational Euler equations are obtained:

$$\delta\dot{\lambda}_u = -\delta\lambda_x \tag{84}$$

$$\delta\dot{\lambda}_v = -\delta\lambda_y \tag{85}$$

$$\delta\dot{\lambda}_w = -\delta\lambda_z \tag{86}$$

$$\begin{aligned}\delta\dot{\lambda}_x = \frac{3\mu}{R^7}\Bigg\{&\left[\lambda_u x\left(3R^2 - 5x^2\right) + \lambda_v y\left(R^2 - 5x^2\right) + \lambda_w z\left(R^2 - 5x^2\right)\right]\delta x \\ &+ \left[\lambda_u y\left(R^2 - 5x^2\right) + \lambda_v x\left(R^2 - 5y^2\right) + \lambda_w z(-5xy)\right]\delta y \\ &+ \left[\lambda_u z\left(R^2 - 5x^2\right) + \lambda_v y(-5xz) + \lambda_w x\left(R^2 - 5z^2\right)\right]\delta z\Bigg\} \\ &- \frac{\mu}{R^5}\left[\left(3x^2 - R^2\right)\delta\lambda_u + y(3x)\delta\lambda_v + z(3x)\delta\lambda_w\right]\end{aligned} \tag{87}$$

$$\delta\dot{\lambda}_y = \frac{-3\mu}{R^7}\left\{\left[\lambda_u y\left(R^2 - 5x^2\right) + \lambda_v x\left(R^2 - 5y^2\right) + \lambda_w z(-5xy)\right]\delta x\right.$$
$$+ \left[\lambda_u x\left(R^2 - 5y^2\right) + \lambda_v y\left(3R^2 - 5y^2\right) + \lambda_w z\left(R^2 - 5y^2\right)\right]\delta y$$
$$\left. + \left[\lambda_u x(-5yz) + \lambda_v z\left(R^2 - 5y^2\right) + \lambda_w y\left(R^2 - 5z^2\right)\right]\delta z\right\}$$
$$- \frac{\mu}{R^5}\left[x(3y)\delta\lambda_u + \left(3y^2 - R^2\right)\delta\lambda_v + z(3y)\delta\lambda_w\right] \tag{88}$$

$$\delta\dot{\lambda}_z = -\frac{3\mu}{R^7}\left\{\left[\lambda_u z\left(R^2 - 5x^2\right) + \lambda_v y(-5xz) + \lambda_w x\left(R^2 - 5z^2\right)\right]\delta x\right.$$
$$+ \left[\lambda_u x(-5yz) + \lambda_v z\left(R^2 - 5y^2\right) + \lambda_w y\left(R^2 - 5z^2\right)\right]\delta y$$
$$\left. + \left[\lambda_u x\left(R^2 - 5z^2\right) + \lambda_v y\left(R^2 - 5z^2\right) + \lambda_w z\left(3R^2 - 5z^2\right)\right]\delta z\right\}$$
$$- \frac{\mu}{R^5}\left[x(3z)\delta\lambda_u + y(3z)\delta\lambda_v + \left(3z^2 - R^2\right)\delta\lambda_w\right] \tag{89}$$

and

$$\delta\dot{\lambda}_m = \frac{T}{m^2}\left(\delta\lambda_u \cos\alpha\cos\theta + \delta\lambda_v \cos\alpha\sin\theta + \delta\lambda_w \sin\alpha\right)$$
$$+ \frac{T}{m^2}\left(-\lambda_u \sin\alpha\cos\theta - \lambda_v \sin\alpha\sin\theta + \lambda_w \cos\alpha\right)\delta\alpha$$
$$+ \frac{T}{m^2}\left(-\lambda_u \cos\alpha\sin\theta + \lambda_v \cos\alpha\cos\theta\right)\delta\theta$$
$$- \frac{2T}{m^3}\left(\lambda_u \cos\alpha\cos\theta + \lambda_v \cos\alpha\sin\theta + \lambda_w \sin\alpha\right)\delta m$$
$$\tag{90}$$

As previously discussed, the transversality conditions for the second variational problem for free δx_i are linearized versions of the transversality conditions for free x_i for the Mayer problem. The general form of the transversality conditions is given by (36). Linearizing equations (66) through (72), the following transversality conditions are obtained:

$$\delta\lambda_u\left(t_f\right) = \left\{k_1\left[1 - \sqrt{\frac{\mu}{\bar{R}\left(u_f^2 + v_f^2\right)}}\left(1 - \frac{u_f^2}{u_f^2 + v_f^2}\right)\right] + k_2\, x_f^2\right\}\Delta u_f$$
$$+ \left[\frac{k_1\, u_f\, v_f}{u_f^2 + v_f^2}\sqrt{\frac{\mu}{\bar{R}\, u_f^2 + v_f^2}} + k_2\, x_f\, y_f\right]\Delta v_f$$
$$+ k_2\left(2u_f\, x_f + v_f\, y_f\right)\Delta x_f + k_2\, x_f\, v_f\, \Delta y_f - \dot{\lambda}_{u_f}\, \delta t_f \tag{91}$$

$$\delta\lambda_v\left(t_f\right) = \left[\frac{k_1\, v_f\, u_f}{u_f^2 + v_f^2}\sqrt{\frac{\mu}{\bar{R}\left(u_f^2 + u_f^2\right)}} + k_2\, x_f\, y_f\right]\Delta u_f$$
$$+ \left\{k_1\left[1 - \sqrt{\frac{\mu}{\bar{R}\left(u_f^2 + v_f^2\right)}}\left(1 - \frac{v_f^2}{u_f^2 + v_f^2}\right)\right] + k_2\, y_f^2\right\}\Delta v_f$$
$$+ k_2\, y_f\, u_f\, \Delta x_f + k_2\left(u_f\, x_f + 2v_f\, y_f\right)\Delta y_f - \dot{\lambda}_{v_f}\, \delta t_f \tag{92}$$

$$\delta\lambda_w\left(t_f\right) = k_3\, \Delta w_f - \dot{\lambda}_{w_f}\, \delta t_f \tag{93}$$

$$\delta\lambda_x(t_f) = k_2\left(2u_f x_f + v_f y_f\right)\Delta u_f + k_2 u_f y_f \Delta v_f$$

$$+\left\{k_2 u_f^2 + k_4\left[1 - \frac{\bar{R}}{\sqrt{x_f^2 + y_5^2}}\left(1 - \frac{x_f^2}{x_f^2 + y_f^2}\right)\right]\right\}\Delta x_f$$

$$+\left[k_2 u_f v_f + k_4 x_f y_f \frac{\bar{R}}{\left(x_f^2 + y_f^2\right)^{3/2}}\right]\Delta y_f - \dot{\lambda}_{x_f}\delta t_f \tag{94}$$

$$\delta\lambda_y(t_f) = k_2 v_f x_f \Delta u_f + k_2\left(u_f x_f + 2v_f y_f\right)\Delta v_f$$

$$+\left[k_2 v_f u_f + k_4 x_f y_f \frac{\bar{R}}{\left(x_f^2 + y_f^2\right)^{3/2}}\right]\Delta x_f$$

$$+\left\{k_2 v_f^2 + k_4\left[1 - \frac{\bar{R}}{\sqrt{x_f^2 + y_f^2}}\left(1 - \frac{x_f^2}{x_f^2 + y_f^2}\right)\right]\right\}\Delta y_f$$

$$- \dot{\lambda}_{y_f}\delta t_f \tag{95}$$

$$\delta\lambda_z(t_f) = k_5 \Delta z_f - \dot{\lambda}_{zf}\delta t_f \tag{96}$$

and

$$\Delta\lambda_m t_f = 0 \tag{97}$$

From the previous discussion, the condition for the parameter δt_f for the Bolza problem is equal to the linearized condition for free t_f for the Mayer problem. The general form is given by equation (37) and linearizing (75), the following result is obtained:

$$H + \sum_{j=1}^{7} \left(-\dot{\lambda}_{j_f}\right) \Delta x_{j_f} + \sum_{j=1}^{7} g_{j_f} \Delta\lambda_{j_f} + \frac{\partial H}{\partial \alpha} \Delta\alpha_f + \frac{\partial H}{\partial \theta} \Delta\theta_f = 0 \tag{98}$$

The switching functions for the second-variational method are given by equation (56).

From the previous discussion, the Euler equations for the control variables for the second variational method are linearized versions of the Euler equations for the Mayer problem with the addition of the $\delta\lambda_{n+k}$ terms. The general expression is given by equation (57), and linearizing equations (73) and (74), the following results are obtained:

$$\delta\theta \left(\delta\lambda_\theta + \frac{\partial^2 H}{\partial \theta^2}\right) = -\left[\frac{\partial H}{\partial \theta} + \frac{\partial^2 H}{\partial \alpha \partial \theta} \delta\alpha + \frac{T}{m}\left(-\cos\alpha \sin\theta \delta\lambda_u + \delta\lambda_v \cos\alpha \cos\theta\right) + \frac{T}{m^2}\left(\lambda_u \cos\alpha \sin\theta - \lambda_v \cos\alpha \cos\theta\right)\delta m\right] \tag{99}$$

and

$$\delta\alpha \left(\delta\lambda_\alpha + \frac{\partial^2 H}{\partial \alpha^2}\right) = -\left[\frac{\partial H}{\partial \alpha} + \frac{\partial^2 H}{\partial \alpha \partial \theta} \delta\theta + \frac{T}{m}\left(-\delta\lambda_u \sin\alpha \cos\theta - \delta\lambda_v \sin\alpha \sin\theta + \delta\lambda_w \cos\alpha\right) + \frac{T}{m^2}\left(\lambda_u \sin\alpha \cos\theta + \lambda_v \sin\alpha \sin\theta - \lambda_w \cos\alpha\right) \delta m\right] \tag{100}$$

It is necessary, from the discussion in section 7, to determine the expression for $\delta\lambda_\theta$ and $\delta\lambda_\alpha$. The expression for $\delta\lambda_\theta$ is given in section 7, and it is necessary only to determine the expression for θ_{min}. From the expression

$$\frac{\partial H}{\partial \theta_{min}} = \frac{T}{m}\left(-\lambda_u \cos\alpha \sin\theta_{min} + \lambda_v \cos\alpha \cos\theta_{min}\right) = 0,$$

$$-\lambda_u \cos\alpha \sin\theta_{min} + \lambda_v \cos\alpha \cos\theta_{min} = 0$$

or

$$\tan\theta_{min} = \frac{\lambda_v}{\lambda_u} \tag{101}$$

Since $\frac{\lambda_v}{\lambda_u}$ and $\frac{-\lambda_v}{-\lambda_u}$ will satisfy the above expression, it is now necessary to determine which expression will minimize H. From (101), it is seen that

$$\sin\theta = \frac{\pm\lambda_v}{\sqrt{\lambda_v^2 + \lambda_u^2}}$$

and

$$\cos\theta = \frac{\pm\lambda_u}{\sqrt{\lambda_v^2 + \lambda_u^2}}$$

If the values $\sin\theta = \frac{\lambda_v}{\sqrt{\lambda_v^2 + \lambda_u^2}}$ and $\cos\theta = \frac{\lambda_u}{\sqrt{\lambda_v^2 + \lambda_u^2}}$ are substituted into equation (48) and the terms not involving θ are dropped since they are constants, the following result is obtained:

$$\lambda_u \frac{\lambda_u}{\sqrt{\lambda_v^2 + \lambda_u^2}} + \lambda_v \frac{\lambda_v}{\sqrt{\lambda_v^2 + \lambda_u^2}} = \frac{\lambda_u^2 + \lambda_v^2}{\sqrt{\lambda_v^2 + \lambda_u^2}}$$

which is a positive quantity.

If the values $\sin\theta = \frac{-\lambda_v}{\sqrt{\lambda_v^2 + \lambda_u^2}}$ and $\cos\theta = \frac{-\lambda_u}{\sqrt{\lambda_v^2 + \lambda_u^2}}$

are used in the same expression, the following result is obtained:

$$-\lambda_v \frac{\lambda_v}{\sqrt{\lambda_v^2 + \lambda_u^2}} - \lambda_u \frac{\lambda_u}{\sqrt{\lambda_v^2 + \lambda_u^2}} = -\left(\frac{\lambda_v^2}{\sqrt{\lambda_v^2 + \lambda_u^2}} + \frac{\lambda_u^2}{\sqrt{\lambda_v^2 + \lambda_u^2}}\right)$$

which is a negative quantity. Therefore, since it is desired to minimize H, the correct value for θ is

$$\theta_{min} = \tan^{-1} \frac{-\lambda_v}{-\lambda_u} \tag{102}$$

A similar analysis must be performed for α_{min}. From the expressions

$$\frac{\partial H}{\partial \alpha_{min}} = \frac{T}{m}\left(-\lambda_u \sin\alpha_{min} \cos\theta - \lambda_v \sin\alpha_{min} \sin\theta + \lambda_w \cos\alpha_{min}\right) = 0,$$

$$-\lambda_u \sin\alpha_{min} \cos\theta - \lambda_v \sin\alpha_{min} \sin\theta + \lambda_w \cos\alpha_{min} = 0$$

or

$$\tan\alpha_{min} = \frac{\lambda_w}{\left(\lambda_u \cos\theta + \lambda_v \sin\theta\right)} \tag{103}$$

Since $\frac{-\lambda_w}{-\left(\lambda_u \cos\theta + \lambda_v \sin\theta\right)}$ and $\frac{\lambda_w}{\lambda_u \cos\theta + \lambda_v \sin\theta}$ will satisfy

the above expression, it is necessary to determine which expression will minimize H. From equation (103) it is seen that

$$\sin\alpha = \frac{\pm\lambda_w}{\sqrt{\lambda_w^{\,2} + \left(\lambda_u \cos\theta + \lambda_v \sin\theta\right)^2}}$$

and

$$\cos\alpha = \frac{\pm\left(\lambda_u \cos\theta + \lambda_v \sin\theta\right)}{\sqrt{\lambda_w^{\,2} + \left(\lambda_u \cos\theta + \lambda_v \sin\theta\right)^2}}$$

If the values

$$\sin\alpha = \frac{\lambda_w}{\sqrt{\lambda_w^{\,2} + \left(\lambda_u \cos\theta + \lambda_v \sin\theta\right)^2}}$$

and

$$\cos\alpha = \frac{\lambda_u \cos\theta + \lambda_v \sin\theta}{\sqrt{\lambda_w^{\,2} + \left(\lambda_u \cos\theta + \lambda_w \sin\theta\right)^2}}$$

are substituted into the expression for H given in section 8 and the terms not involving α are dropped since they are constants, the following result is obtained:

$$\frac{\lambda_u \cos\theta\left(\lambda_u \cos\theta + \lambda_v \sin\theta\right) + \lambda_v \sin\theta\left(\lambda_u \cos\theta + \lambda_v \sin\theta\right) + \lambda_w^{\,2}}{\sqrt{\lambda_w^{\,2} + \left(\lambda_u \cos\theta + \lambda_v \sin\theta\right)^2}}$$

$$= \frac{\lambda_u^{\,2} \cos^2\theta + 2\lambda_v\lambda_u \sin\theta\cos\theta + \lambda_v^{\,2} \sin^2\theta + \lambda_w^{\,2}}{\sqrt{\lambda_w^{\,2} + \left(\lambda_u \cos\theta + \lambda_v \sin\theta\right)^2}}$$

$$= \frac{\left(\lambda_u \cos\theta + \lambda_v \sin\theta\right)^2 + \lambda_w^{\,2}}{\sqrt{\lambda_w^{\,2} + \left(\lambda_u \cos\theta + \lambda_v \sin\theta\right)^2}}$$

which is a positive quantity.

If the values $\sin\alpha = \dfrac{-\lambda_w}{\sqrt{\lambda_w^2 + \lambda_u \cos\theta + \lambda_v \sin\theta^{2}}}$ and

$\cos\alpha = -\dfrac{\left(\lambda_u \cos\theta + \lambda_v \sin\theta\right)}{\sqrt{\lambda_w^2 + \left(\lambda_u \cos\theta + \lambda_v \sin\theta\right)^2}}$ are used in the same expression,

the following result is obtained:

$$-\frac{\lambda_u \cos\theta \left(\lambda_u \cos\theta + \lambda_v \sin\theta\right) - \lambda_v \sin\theta \left(\lambda_u \cos\theta + \lambda_v \sin\theta\right) - \lambda_w^2}{\sqrt{\lambda_w^2 + \left(\lambda_u \cos\theta + \lambda_v \sin\theta\right)^2}}$$

which is a negative quantity.

As a result of this analysis and using θ_{min} as the correct value of θ, the following result is obtained for the desired α

$$\alpha_{min} = \tan^{-1}\left[\frac{-\lambda_w}{-\left(\lambda_u \cos\theta_{min} + \lambda_v \sin\theta_{min}\right)}\right] \qquad (104)$$

13. METHOD OF SOLUTION FOR PENALTY FUNCTION PROCESS

The previously defined results have yielded sifficient conditions to solve the penalty function process. The following discussion is a step-by-step description of the method of solution. This memorandum is not intended to describe the actual program and a succeeding memorandum will discuss the program in detail. The sequence of calculations is for the problem discussed in section 12. Solution techniques for other problems require only relatively minor changes.

Step 1. Store the first estimate for $\theta(t)$, $\alpha(t)$, t_1, and t_2.

Step 2. Integrate numerically systems (41) through (47), employing the first estimate of $\alpha(t)$, $\theta(t)$, t_1 and t_2 from Step 1.

Step 3. Terminate the trajectory at a time t_f such that P' attains a minimum. The conditions for this termination are discussed in section 6. Since the problem was defined for fixed-state variables at initial time, the initial values of the state variables are the constants of integration. This trajectory is the reference trajectory and variations in the control variables and switching times will be determined which will minimize P' on the succeeding trajectory.

Step 4. The numerical calculations are now at t_f and Lagrange multipliers at t_f can be calculated from the relations (66) through (72). Using these values as constants of integration, the adjoint system given by equations (59) through (65) can be integrated backward in time to time t_o to obtain initial values of the Lagrange multipliers.

Step 5. It is necessary to determine a transition matrix to calculate changes in final values of the Lagrange multipliers as a result of changes in initial values of the adjoint variables and changes in t_1 and t_2. The following expressions are used to form the transition matrix:

$$\delta\lambda_i\left(t_f\right) = \sum_{j=1}^{7} \frac{\partial\lambda_i\left(t_f\right)}{\partial\lambda_j\left(t_o\right)}\,\delta\lambda_j\left(t_o\right) + \frac{\partial\lambda_i\left(t_f\right)}{\partial t_1}\,\delta t_1 + \frac{\partial\lambda_i\left(t_f\right)}{\partial t_2}\,\delta t_2 + \hat{\lambda}_i, \qquad i = 1, \ldots, 7 \tag{105}$$

$$\delta x_i\left(t_f\right) = \sum_{j=1}^{7} \frac{\partial x_i\left(t_f\right)}{\partial\lambda_j\left(t_o\right)}\,\delta\lambda_j\left(t_o\right) + \frac{\partial x_i\left(t_f\right)}{\partial t_1}\,\delta t_1 + \frac{\partial x_i\left(t_f\right)}{\partial t_2}\,\delta t_2 + \hat{x}_i, \qquad i = 1, \ldots, 7 \tag{106}$$

The hat terms in the previous expressions are a result of the nonhomogeneous terms $\frac{\partial H}{\partial\alpha}$ and $\frac{\partial H}{\partial\theta}$, appearing in the equation for $\delta\alpha$ and $\delta\theta$ as given by equations (99) and (100). If these nonhomogeneous terms are nonzero, $\delta\lambda_i\left(t_f\right)$, $i = 1, \ldots, 7$, and $\delta x_i\left(t_f\right)$, $i = 1, \ldots, 7$ may be nonzero even if all of the values $\delta\lambda_i\left(t_o\right)$, $i = 1, \ldots, 7$ and

δt_1 and δt_2 values are zero. The above system can be written in the matrix form shown on the following page. It is necessary to evaluate the 14×9 matrix of partial derivatives and the 14×1 matrix of hat terms. A total of 10 separate calculations are necessary for this operation and are listed below. The values of $\delta\lambda_\theta$ and $\delta\lambda_\alpha$ are calculated at every integration point using the method discussed in section 9.

Step a. Set δt_1, δt_2, and $\delta\lambda_i\left(t_f\right)$, $i = 1, \ldots, 7$ values to zero. Integrate equation (41) through (47), equations (59) through (65), equations (76) through (82), and equations (84) through (90) to time t_f using $\delta\alpha$ and $\delta\theta$ given by the equations given on page 50. Since all variations are set to zero for this operation, the values obtained for $\delta\lambda_i(t)$ and $\delta x_i(t)$, $i = 1, \ldots, 7$, are the effects of the nonhomogeneous terms in $\delta\alpha$ and $\delta\theta$, and the $\delta\lambda_i\left(t_f\right)$ and $\delta x_i\left(t_f\right)$, $i = 1, \ldots, 7$ values are the terms of the 14×1 hat matrix.

For steps b through j the terms $\frac{\partial H}{\partial \alpha}$ and $\frac{\partial H}{\partial \theta}$ in the $\delta\alpha$ and $\delta\theta$ equations are set to zero since step a determines the effect of these terms.

Step b. Set $\delta t_1 = 0$, $\delta t_2 = 0$, $\delta\lambda_u\left(t_o\right) = 1$, and $\delta\lambda_i = 0$,

$i = 2, \ldots, 7.$

Integrating the same equations listed in step a forward to t_f, the $\delta x_i\left(t_f\right)$ and $\delta\lambda_i\left(t_f\right)$, $i = 1, \ldots, 7$ values obtained represent changes in the state variables and in the Lagrange multipliers at final time due to a unit change in the initial value of the λ_w multiplier. These values represent column 1 in the 14×9 transition matrix.

Step c. Same operation as step b except set $\delta\lambda_v\left(t_o\right) = 1$, $\delta\lambda_i\left(t_o\right) = 0$, $i = 1, 3, \ldots, 7$ to calculate the second column in the 14×9 transition matrix.

Step d. Same operation as step b except set $\delta\lambda_w\left(t_o\right) = 1$ to calculate third column in 14×9 transition matrix.

Step e. Same operation as step b except set $\delta\lambda_x\left(t_o\right) = 1$ to calculate fourth column in 14×9 transition matrix.

$$
\begin{bmatrix} \delta\lambda_u(t_f) \\ \delta\lambda_v(t_f) \\ \delta\lambda_w(t_f) \\ \delta\lambda_x(t_f) \\ \delta\lambda_y(t_f) \\ \delta\lambda_z(t_f) \\ \delta\lambda_m(t_f) \\ \delta u(t_f) \\ \delta v(t_f) \\ \delta w(t_f) \\ \delta x(t_f) \\ \delta y(t_f) \\ \delta z(t_f) \\ \delta m(t_f) \end{bmatrix}
=
\begin{bmatrix}
\frac{\partial\lambda_u(t_f)}{\partial\lambda_u(t_o)} & \frac{\partial\lambda_u(t_f)}{\partial\lambda_v(t_o)} & \frac{\partial\lambda_u(t_f)}{\partial\lambda_w(t_o)} & \frac{\partial\lambda_u(t_f)}{\partial\lambda_x(t_o)} & \frac{\partial\lambda_u(t_f)}{\partial\lambda_y(t_o)} & \frac{\partial\lambda_u(t_f)}{\partial\lambda_z(t_o)} & \frac{\partial\lambda_u(t_f)}{\partial\lambda_m(t_o)} & \frac{\partial\lambda_u(t_f)}{\partial t_1} & \frac{\partial\lambda_u(t_f)}{\partial t_2} \\
\frac{\partial\lambda_v(t_f)}{\partial\lambda_u(t_o)} & \frac{\partial\lambda_v(t_f)}{\partial\lambda_v(t_o)} & \frac{\partial\lambda_v(t_f)}{\partial\lambda_w(t_o)} & \frac{\partial\lambda_v(t_f)}{\partial\lambda_x(t_o)} & \frac{\partial\lambda_v(t_f)}{\partial\lambda_y(t_o)} & \frac{\partial\lambda_v(t_f)}{\partial\lambda_z(t_o)} & \frac{\partial\lambda_v(t_f)}{\partial\lambda_m(t_o)} & \frac{\partial\lambda_v(t_f)}{\partial t_1} & \frac{\partial\lambda_v(t_f)}{\partial t_2} \\
\frac{\partial\lambda_w(t_f)}{\partial\lambda_u(t_o)} & \frac{\partial\lambda_w(t_f)}{\partial\lambda_v(t_o)} & \frac{\partial\lambda_w(t_f)}{\partial\lambda_w(t_o)} & \frac{\partial\lambda_w(t_f)}{\partial\lambda_x(t_o)} & \frac{\partial\lambda_w(t_f)}{\partial\lambda_y(t_o)} & \frac{\partial\lambda_w(t_f)}{\partial\lambda_z(t_o)} & \frac{\partial\lambda_w(t_f)}{\partial\lambda_m(t_o)} & \frac{\partial\lambda_w(t_f)}{\partial t_1} & \frac{\partial\lambda_w(t_f)}{\partial t_2} \\
\frac{\partial\lambda_x(t_f)}{\partial\lambda_u(t_o)} & \frac{\partial\lambda_x(t_f)}{\partial\lambda_v(t_o)} & \frac{\partial\lambda_x(t_f)}{\partial\lambda_w(t_o)} & \frac{\partial\lambda_x(t_f)}{\partial\lambda_x(t_o)} & \frac{\partial\lambda_x(t_f)}{\partial\lambda_y(t_o)} & \frac{\partial\lambda_x(t_f)}{\partial\lambda_z(t_o)} & \frac{\partial\lambda_x(t_f)}{\partial\lambda_m(t_o)} & \frac{\partial\lambda_x(t_f)}{\partial t_1} & \frac{\partial\lambda_x(t_f)}{\partial t_2} \\
\frac{\partial\lambda_y(t_f)}{\partial\lambda_u(t_o)} & \frac{\partial\lambda_y(t_f)}{\partial\lambda_v(t_o)} & \frac{\partial\lambda_y(t_f)}{\partial\lambda_w(t_o)} & \frac{\partial\lambda_y(t_f)}{\partial\lambda_x(t_o)} & \frac{\partial\lambda_y(t_f)}{\partial\lambda_y(t_o)} & \frac{\partial\lambda_y(t_f)}{\partial\lambda_z(t_o)} & \frac{\partial\lambda_y(t_f)}{\partial\lambda_m(t_o)} & \frac{\partial\lambda_y(t_f)}{\partial t_1} & \frac{\partial\lambda_y(t_f)}{\partial t_2} \\
\frac{\partial\lambda_z(t_f)}{\partial\lambda_u(t_o)} & \frac{\partial\lambda_z(t_f)}{\partial\lambda_v(t_o)} & \frac{\partial\lambda_z(t_f)}{\partial\lambda_w(t_o)} & \frac{\partial\lambda_z(t_f)}{\partial\lambda_x(t_o)} & \frac{\partial\lambda_z(t_f)}{\partial\lambda_y(t_o)} & \frac{\partial\lambda_z(t_f)}{\partial\lambda_z(t_o)} & \frac{\partial\lambda_z(t_f)}{\partial\lambda_m(t_o)} & \frac{\partial\lambda_z(t_f)}{\partial t_1} & \frac{\partial\lambda_z(t_f)}{\partial t_2} \\
\frac{\partial\lambda_m(t_f)}{\partial\lambda_u(t_o)} & \frac{\partial\lambda_m(t_f)}{\partial\lambda_v(t_o)} & \frac{\partial\lambda_m(t_f)}{\partial\lambda_w(t_o)} & \frac{\partial\lambda_m(t_f)}{\partial\lambda_x(t_o)} & \frac{\partial\lambda_m(t_f)}{\partial\lambda_y(t_o)} & \frac{\partial\lambda_m(t_f)}{\partial\lambda_z(t_o)} & \frac{\partial\lambda_m(t_f)}{\partial\lambda_m(t_o)} & \frac{\partial\lambda_m(t_f)}{\partial t_1} & \frac{\partial\lambda_m(t_f)}{\partial t_2} \\
\frac{\partial u(t_f)}{\partial\lambda_u(t_o)} & \frac{\partial u(t_f)}{\partial\lambda_v(t_o)} & \frac{\partial u(t_f)}{\partial\lambda_w(t_o)} & \frac{\partial u(t_f)}{\partial\lambda_x(t_o)} & \frac{\partial u(t_f)}{\partial\lambda_y(t_o)} & \frac{\partial u(t_f)}{\partial\lambda_z(t_o)} & \frac{\partial u(t_f)}{\partial\lambda_m(t_o)} & \frac{\partial u(t_f)}{\partial t_1} & \frac{\partial u(t_f)}{\partial t_2} \\
\frac{\partial v(t_f)}{\partial\lambda_u(t_o)} & \frac{\partial v(t_f)}{\partial\lambda_v(t_o)} & \frac{\partial v(t_f)}{\partial\lambda_w(t_o)} & \frac{\partial v(t_f)}{\partial\lambda_x(t_o)} & \frac{\partial v(t_f)}{\partial\lambda_y(t_o)} & \frac{\partial v(t_f)}{\partial\lambda_z(t_o)} & \frac{\partial v(t_f)}{\partial\lambda_m(t_o)} & \frac{\partial v(t_f)}{\partial t_1} & \frac{\partial v(t_f)}{\partial t_2} \\
\frac{\partial w(t_f)}{\partial\lambda_u(t_o)} & \frac{\partial w(t_f)}{\partial\lambda_v(t_o)} & \frac{\partial w(t_f)}{\partial\lambda_w(t_o)} & \frac{\partial w(t_f)}{\partial\lambda_x(t_o)} & \frac{\partial w(t_f)}{\partial\lambda_y(t_o)} & \frac{\partial w(t_f)}{\partial\lambda_z(t_o)} & \frac{\partial w(t_f)}{\partial\lambda_m(t_o)} & \frac{\partial w(t_f)}{\partial t_1} & \frac{\partial w(t_f)}{\partial t_2} \\
\frac{\partial x(t_f)}{\partial\lambda_u(t_o)} & \frac{\partial x(t_f)}{\partial\lambda_v(t_o)} & \frac{\partial x(t_f)}{\partial\lambda_w(t_o)} & \frac{\partial x(t_f)}{\partial\lambda_x(t_o)} & \frac{\partial x(t_f)}{\partial\lambda_y(t_o)} & \frac{\partial x(t_f)}{\partial\lambda_z(t_o)} & \frac{\partial x(t_f)}{\partial\lambda_m(t_o)} & \frac{\partial x(t_f)}{\partial t_1} & \frac{\partial x(t_f)}{\partial t_2} \\
\frac{\partial y(t_f)}{\partial\lambda_u(t_o)} & \frac{\partial y(t_f)}{\partial\lambda_v(t_o)} & \frac{\partial y(t_f)}{\partial\lambda_w(t_o)} & \frac{\partial y(t_f)}{\partial\lambda_x(t_o)} & \frac{\partial y(t_f)}{\partial\lambda_y(t_o)} & \frac{\partial y(t_f)}{\partial\lambda_z(t_o)} & \frac{\partial y(t_f)}{\partial\lambda_m(t_o)} & \frac{\partial y(t_f)}{\partial t_1} & \frac{\partial y(t_f)}{\partial t_2} \\
\frac{\partial z(t_f)}{\partial\lambda_u(t_o)} & \frac{\partial z(t_f)}{\partial\lambda_v(t_o)} & \frac{\partial z(t_f)}{\partial\lambda_w(t_o)} & \frac{\partial z(t_f)}{\partial\lambda_x(t_o)} & \frac{\partial z(t_f)}{\partial\lambda_y(t_o)} & \frac{\partial z(t_f)}{\partial\lambda_z(t_o)} & \frac{\partial z(t_f)}{\partial\lambda_m(t_o)} & \frac{\partial z(t_f)}{\partial t_1} & \frac{\partial z(t_f)}{\partial t_2} \\
\frac{\partial m(t_f)}{\partial\lambda_u(t_o)} & \frac{\partial m(t_f)}{\partial\lambda_v(t_o)} & \frac{\partial m(t_f)}{\partial\lambda_w(t_o)} & \frac{\partial m(t_f)}{\partial\lambda_x(t_o)} & \frac{\partial m(t_f)}{\partial\lambda_y(t_o)} & \frac{\partial m(t_f)}{\partial\lambda_z(t_o)} & \frac{\partial m(t_f)}{\partial\lambda_m(t_o)} & \frac{\partial m(t_f)}{\partial t_1} & \frac{\partial m(t_f)}{\partial t_2}
\end{bmatrix}
\begin{bmatrix} \delta\lambda_u(t_o) \\ \delta\lambda_v(t_o) \\ \delta\lambda_w(t_o) \\ \delta\lambda_x(t_o) \\ \delta\lambda_y(t_o) \\ \delta\lambda_z(t_o) \\ \delta\lambda_m(t_o) \\ \delta t_1 \\ \delta t_2 \end{bmatrix}
+
\begin{bmatrix} \hat\lambda_u(t_f) \\ \hat\lambda_v(t_f) \\ \hat\lambda_w(t_f) \\ \hat\lambda_x(t_f) \\ \hat\lambda_y(t_f) \\ \hat\lambda_z(t_f) \\ \hat\lambda_m(t_f) \\ \hat u(t_f) \\ \hat v(t_f) \\ \hat w(t_f) \\ \hat x(t_f) \\ \hat y(t_f) \\ \hat z(t_f) \\ \hat m(t_f) \end{bmatrix}
$$

Transition Matrix

Step f. Same operation as step b except set $\delta\lambda_y(t_o) = 1$ to calculate fifth column in 14×9 transition matrix.

Step g. Same operation as step b except set $\delta\lambda_z(t_o) = 1$ to calculate sixth column in 14×9 transition matrix.

Step h. Same operation as step b except set $\delta\lambda_m(t_o) = 1$ to calculate seventh column in the 14×9 transition matrix.

Step i. Set $\delta\lambda_i(t_o) = 0$, $i = 1, \ldots, 7$, $\delta t_2 = 0$, and $\delta t_1 = 1$. Integrate the equations listed in step a forward to time t_1. At this point the state variables should be jumped by the expressions calculated in section 10. That is,

$$\delta u = \left(\frac{T^+ - T^-}{m} \cos\theta \cos\alpha\right)\delta t_1$$

$$\delta_v = \left(\frac{T^+ - T^-}{m} \sin\theta \cos\alpha\right)\delta t_1$$

$$\delta w = \left(\frac{T^+ - T^-}{m} \sin\alpha\right)\delta t_1$$

$$\delta x = 0$$

$$\delta y = 0$$

$$\delta z = 0$$

and

$$\delta m = \frac{-\left(T^+ - T^-\right)}{c}\delta t_1$$

where

$$\delta t_1 = 1$$

Integrating the system from t_1 to t_f yields the desired result for column 8 in the 14×9 transition matrix.

Step 6. The condition for the parameter δt_f given by equation (98), equation (56) evaluated at t_1 and t_2, the set of equations given by

(105), the set of equations given by (91) through (97), and the set of equations given by (106), yield a set of 24 equations and 24 unknowns. The 24 unknowns are as follows:

$$\delta\lambda_i(t_o), \qquad i = 1, \ldots, 7$$

$$\delta\lambda_i(t_f), \qquad i = 1, \ldots, 7$$

$$\delta x_i(t_f), \qquad i = 1, \ldots, 7$$

$$\delta t_1$$

$$\delta t_2$$

and

$$\delta t_f$$

The coefficients of the unknows except equation (56) are evaluated at final time.

The coefficients of equations (56) for δt_1 are evaluated at t_1 in step 5(i) above. The terms $\delta\alpha(t_1)$, $\delta\theta(t_1)$, and $\delta\lambda(t_1)$, $i = 1, \ldots, 7$, are evaluated from the following expressions:

$$\delta\alpha(t_1) = \sum_{i=1}^{7} \frac{\partial\alpha(t_1)}{\partial\lambda_i(t_o)}\delta\lambda_i(t_o) + \hat{\alpha}(t_1)$$

$$\delta\theta(t_1) = \sum_{i=1}^{7} \frac{\partial\theta(t_1)}{\partial\lambda_i(t_o)}\delta\lambda_i(t_o) + \hat{\theta}(t_1)$$

and

$$\delta\lambda_i(t_i) = \sum_{j=1}^{7} \frac{\partial\lambda_i(t_1)}{\partial\lambda_j(t_o)}\delta\lambda_j(t_o) + \hat{\lambda}(t_1), \qquad i = 1, \ldots, 7$$

The coefficients of equation (56) for δt_2 determination are evaluated at t_2 in step 5 above. The terms $\delta\alpha(t_2)$, $\delta\theta(t_2)$, and $\delta\lambda_i(t_2)$, $i = 1, \ldots, 7$, are evaluated from the following expression:

$$\delta\alpha(t_2) = \sum_{i=1}^{7} \frac{\partial\alpha(t_2)}{\partial\lambda_i(t_o)} \delta\lambda_i(t_o) + \frac{\partial\alpha(t_2)}{\partial t_1} + \hat{\alpha}(t_2)$$

$$\delta\theta(t_2) = \sum_{i=1}^{7} \frac{\partial\theta(t_2)}{\partial\lambda_i(t_o)} \delta\lambda_i(t_o) + \frac{\partial\theta(t_2)}{\partial t_1} + \hat{\theta}(t_2)$$

and

$$\delta\lambda_i(t_2) = \sum_{j=1}^{7} \frac{\partial\lambda_i(t_2)}{\partial\lambda_j(t_o)} \delta\lambda_j(t_o) + \frac{\partial\theta(t_2)}{\partial t_1} \delta t_1 + \hat{\lambda}_i(t_2), \qquad i = 1 \ldots, 7$$

The set of 24 equations can be solved simultaneously to determine the 24 unknowns.

Step 7. With the δt_1, δt_2, and $\delta\lambda_i(t_o)$, $i = 1, \ldots, 7$, values determined in step 6, integrate equations listed in step 5(a) forward in time to calculate $\delta\alpha(t)$ and $\delta\theta(t)$ given by the equations for $\delta\alpha$ and $\delta\theta$ given by equations (99) and (100). At t_1 the variational state equations must be jumped by the following relationships:

$$\delta u(t_1) = \left(\frac{T^+ - T^-}{m} \cos\theta \cos\alpha\right)_{t_1} \delta t_1$$

$$\delta v(t_1) = \left(\frac{T^+ - T^-}{m} \sin\theta \cos\alpha\right)_{t_1} \delta t_1$$

$$\delta w(t_1) = \left(\frac{T^+ - T^-}{m} \sin\alpha\right)_{t_1} \delta t_1$$

where δt_1 is determined in step 6. A similar jump is made at t_2 in the state variables where

$$\delta u(t_2) = \left(\frac{T^+ - T^-}{m} \cos\theta \cos\alpha\right)_{t_2} \delta t_2$$

$$\delta v(t_2) = \left(\frac{T^+ - T^-}{m} \sin\theta \cos\alpha\right)_{t_2} \delta t_2$$

$$\delta w(t_2) = \left(\frac{T^+ - T^-}{m} \sin\alpha\right)_{t_2} \delta t_2$$

and

$$\delta m(t_2) = -\left(\frac{T^+ - T^-}{c}\right)_{t_1} \delta t_1$$

where δt_2 is determined from step 6.

Step 8. The values of δt_1 and δt_2 calculated in step 6 and the values of $\delta\alpha(t)$ and $\delta\theta(t)$ calculated in step 7 are added to the values of t_1, t_2, $\alpha(t)$, and $\theta(t)$ which were stored in step 1. The entire process from step 2 through step 8, using the new switching times and control variables, is now repeated. The new value of P' calculated in step 3 should be smaller than the previous value. This entire process is repeated until improvement in the function P' becomes small or until the program converges to a minimum of P'.

14. REFINEMENT PROCESS

Since the penalty function process approximation discussed in the preceding section will converge to a solution whose terminal values differ from those prescribed, a refinement process is necessary to allow complete convergence. As in reference 2, it is assumed that the required end conditions are of the following form:

$$\psi_i = x_{i_f} - \tilde{x}_i = 0, \qquad i = 1, \ldots, a, \tag{107}$$

where

$\tilde{x}_i$ = desired value of x_{i_f}

This particular type of end condition greatly simplifies the resulting equations and will demonstrate the required result. The function to be minimized is of the form (3), subject to the end conditions (2). Except for the transversality conditions all conditions are the same as the original problem. It is necessary to discuss the form of the new transversality conditions.

An expansion of the function P similar to that given by (28) for the function P' is as follows:

$$P = P\left(\bar{x}_{a+1_f}, \ . \ . \ . , \bar{x}_{n_f}, t_f\right) + \sum_{i=a+1}^{n} \frac{\partial P}{\partial x_{i_f}} \Delta x_{i_f}$$

$$+ \frac{\partial P}{\partial t_f} \delta t_f + \frac{1}{2} \frac{\partial^2 P}{\partial t_f^{\,2}} \delta t_f^{\,2} + \frac{1}{2} \sum_{i=a+1}^{n} \sum_{j=a+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial x_{j_f}} \Delta x_{i_f} \Delta x_{j_f}$$

$$+ \sum_{i=a+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial t_f} \Delta x_{i_f} \delta t_f \tag{108}$$

The fixed-state variable terminal conditions are

$$\bar{x}_{i_f} + \Delta x_{i_f} - \tilde{x}_{i_f} = 0, \qquad i = 1, \ . \ . \ . , a, \tag{109}$$

where

$\bar{x}_{i_f}$ = reference trajectory values

Δx_{i_f} = second order approximations to terminal value increments given by (27).

The constraints (107) may be adjoined to (108)by means of constant multipliers μ_i, $i = 1, \ . \ . \ . , a$. Approximating the μ_i to zero and first order terms where $\bar{\mu}_i$ is the value for the reference trajectory, and applying the multiplier rule, the following second order approximation to P is obtained:

$$
\begin{aligned}
J_0 + J_1 + \frac{1}{2} J_2 = P\left(\bar{x}_{a+1_f}, \ldots, \bar{x}_{n_f}, \bar{t}_f\right) \\
& + \sum_{i=a+1}^{n} \frac{\partial P}{\partial x_{i_f}} \Bigg[\xi_{i_f} + \Bigg(\bar{g}_{i_f} + \sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{i_f}} \delta x_{j_f} \\
& + \sum_{k=1}^{m} \frac{\partial g_{i_f}}{\partial y_{k_f}} \delta y_{k_f} \Bigg) \delta t_f + \frac{1}{2} \Bigg(\sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \bar{g}_{j_f} \\
& + \sum_{k=1}^{m} \frac{\partial g_{i_f}}{\partial y_{k_f}} \dot{y}_{k_f} + \frac{\partial g_{i_f}}{\partial t_f} \Bigg) \delta t_f^{\,2} \Bigg] + \frac{\partial P}{\partial t_f} \delta t_f \\
& + \frac{1}{2} \sum_{i=a+1}^{n} \sum_{j=a+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial x_{j_f}} \left(\delta x_{i_f} + \bar{g}_{i_f} \delta t_f \right) \left(\delta x_{j_f} + \bar{g}_{j_f} \delta t_f \right) \\
& + \sum_{i=a+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial t_f} \left(\delta x_{i_f} + \bar{g}_{i_f} \delta t_f \right) \delta t_f + \frac{1}{2} \frac{\partial^2 P}{\partial t_f^{\,2}} \delta t_f^{\,2} \\
& + \sum_{i=1}^{a} \bar{\mu}_1 \Bigg[\xi_{i_f} + \Bigg(\bar{g}_{i_f} + \sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta x_{j_f} + \sum_{k=1}^{m} \frac{\partial g_{i_f}}{\partial y_{k_f}} \delta y_{k_f} \Bigg) \delta t_f \\
& + \frac{1}{2} \Bigg(\sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \bar{g}_{j_f} + \sum_{k=1}^{m} \frac{\partial g_{i_f}}{\partial y_{k_f}} \dot{y}_{k_f} + \frac{\partial g_{i_f}}{\partial t_f} \Bigg) \delta t_f^{\,2} \\
& + \bar{x}_{i_f} - \tilde{x}_{i_f} \Bigg] + \sum_{i=1}^{a} \delta \mu_i \left(\delta x_{i_f} + \bar{g}_{i_f} \delta t_f + \bar{x}_{j_f} - \tilde{x}_{i_f} \right)
\end{aligned}
\tag{110}
$$

The transversality condition for the state variable δx_{i_f} is of the following form:

$$\delta\lambda_{j_f} = \frac{\partial\left(J_o + J_1 + J_2\right)}{\partial x_j}$$

Taking the indicated partials of (110) the following expresions are obtained:

$$\delta\lambda_{j_f} = \sum_{i=a+1}^{n} \frac{\partial P}{\partial x_{i_f}} \frac{\partial g_{i_f}}{\partial x_{i_f}} \delta t_f + \sum_{i=1}^{a} \bar{\mu}_i \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta t_f$$

$$+ \delta\mu_j, \qquad j = 1, \ldots, a \tag{111}$$

and

$$\sum_{i=a+1}^{n} \frac{\partial P}{\partial x_{i_f}} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta t_f + \sum_{i=1}^{a} \bar{\mu}_i \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta t_f$$

$$+ \sum_{i=a+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial x_{j_f}} \left(\delta x_{i_f} + g_{i_f} \delta t_f\right)$$

$$+ \frac{\partial^2 P}{\partial x_{j_f} \partial t_f} - \delta\lambda_{j_f} = 0, \qquad j = a + 1, \ldots, n$$

$$\tag{112}$$

For the parameter δt_f, the required condition is

$$\frac{\partial\left(J_o + J_1 + J_2\right)}{\partial t_f} = 0 \tag{113}$$

or

$$\sum_{i=a+1}^{n} \frac{\partial P}{\partial x_{i_f}} \left[\bar{g}_{i_f} + \sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta x_{j_f} + \sum_{k=1}^{m} \frac{\partial g_{i_f}}{\partial y_{k_f}} \delta y_{k_f} \right.$$

$$\left. + \left(\sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_j} g_{j_f} + \sum_{k=1}^{n} \frac{\partial g_{i_f}}{\partial y_{k_f}} \dot{y}_{k_f} + \frac{\partial g_{i_f}}{\partial t_f} \right) \delta t_f \right]$$

$$+ \frac{\partial P}{\partial t_f} + \sum_{i=a+1}^{n} \sum_{j=a+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial x_{j_f}} \left(\delta x_{i_f} + g_{i_f} \delta t_f \right) \bar{g}_{j_f}$$

$$+ \sum_{i=a+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial t_f} \left(\delta x_{i_f} + 2 g_{i_f} \delta t_f \right) + \frac{\partial^2 P}{\partial t_f^{\,2}} \delta t_f$$

$$+ \sum_{i=1}^{a} \mu_i \left[g_{i_f} + \sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta x_{j_f} \right.$$

$$\left. + \sum_{k=1}^{a} \frac{\partial g_{i_f}}{\partial y_{k_f}} \delta y_{k_f} + \left(\sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} g_{j_f} + \sum_{k=1}^{n} \frac{\partial g_{i_f}}{\partial y_{k_f}} \dot{y}_{k_f} + \frac{\partial g_{i_f}}{\partial t_f} \right) \delta t_f \right]$$

$$+ \sum_{i=1}^{a} \delta \mu_i \bar{g}_{i_f} = 0 \tag{114}$$

For the $\delta\mu_j$ value, the required condition is

$$\frac{\partial \left(J_o + J_1 + J_2 \right)}{\partial \mu_j} = 0, \qquad j = 1, \ldots, a$$

or

$$\delta x_{j_f} + \bar{g}_{j_f} \delta t_f + \bar{x}_{j_f} - \tilde{x}_{j_f} = 0, \qquad j = 1, \ldots, m \tag{115}$$

The problem is normally treated with the augmented function

$$P^* = P + \mu_i \left(x_i - \tilde{x}_i\right) \qquad i = 1, \ldots, a$$

The transversality conditions for x_{i_f} are

$$\frac{\partial P^*}{\partial x_{i_f}} \equiv \lambda_{i_f}, \qquad i = 1, \ldots, n$$

or

$$\mu_i = \lambda_i, \qquad i = 1, \ldots, a \tag{116}$$

and

$$\frac{\partial P}{\partial x_{i_f}} = \lambda_{i_f}, \qquad i = a + 1, \ldots, n \tag{117}$$

Calculating a linearized version of equation (116), the following result is obtained:

$$\lambda_j + \Delta\lambda_j = \mu_j + \Delta\mu_j$$

or

$$\lambda_j + \delta\lambda_j + \dot{\lambda}_j \delta t_f = \mu_j + \delta\mu_j$$

Since $\dot{\lambda}_j = -\frac{\partial H}{\partial x_j}$ and $\mu_j = \lambda_j$, the above equation is written as

$$\delta\lambda_j - \frac{\partial H}{\partial x_j} \delta t_f - \delta\mu_j = 0 \tag{118}$$

Rewriting equation (111), using equation (116) and (117),

$$\delta\lambda_{j_f} = \sum_{i=a+1}^{n} \lambda_{i_f} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta t_f + \sum_{i=1}^{a} \lambda_{i_f} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta t_f$$

$$+ \delta\mu_j, \qquad j = 1, \ldots, a$$

or

$$\delta\lambda_{j_f} = \sum_{i=1}^{n} \lambda_{i_f} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta t_f + \delta\mu_j \qquad j = 1, \ldots, a$$

Since

$$\sum_{i=1}^{n} \lambda_{i_f} \frac{\partial g_{i_f}}{\partial x_j} = \frac{\partial H}{\partial x_j}$$

the above equation can be written as

$$\delta\lambda_{j_f} = \frac{\partial H}{\partial x_j} \delta t_f + \delta\mu_j, \qquad j = 1, \ldots, a \tag{119}$$

Comparing this linearized transversality condition of the original problem with equation (118) it is seen that they are equal.

Calculating a linearized version of equation (117), the following result is obtained:

$$\frac{\partial P}{\partial x_{j_f}} + \sum_{g=a+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial x_{g_f}} \left(\delta x_{g_f} + g_{q_f} \delta t_f\right) + \frac{\partial^2 P}{\partial x_{j_f} \partial t_f} \delta t_f$$

$$- \lambda_{j_f} - \delta\lambda_j + \frac{\partial H}{\partial x_j} \delta t_f = 0. \qquad j = a + 1, \ldots, n$$

(120)

Since $\frac{\partial P}{\partial x_{j_f}} = \lambda_{j_f}$ and using the relations (116), it is seen that equations (112) and (120) are equal.

The transversality condition for free final time is

$$\frac{\partial P}{\partial t_f} + H_f = 0$$

Calculating a linearized version of this equation, the following result is obtained:

$$\frac{\partial P}{\partial t_f} + H_f + \sum_{q=a+1}^{n} \frac{\partial^2 P}{\partial t_f \partial x_{q_f}} \left(\delta x_{q_f} + g_{q_f} \delta t_f\right)$$

$$+ \frac{\partial^2 P}{\partial t_f^{\,2}} \delta t_f + \sum_{j=1}^{n} \left(\delta \lambda_{j_f} - \frac{\partial H}{\partial x_j} \delta t_f\right) g_j$$

$$+ \sum_{i=1}^{n} \lambda_i \left[\sum_{g=1}^{n} \frac{\partial g_i}{\partial x_{g_f}} \left(\delta x_{g_f} + g_{q_f} \delta t_f\right)\right.$$

$$\left. + \sum_{k=1}^{n} \frac{\partial g_i}{\partial y_{k_f}} \left(\delta y_{k_f} + \dot{y}_{k_f} \delta t_f\right) + \frac{\partial g_i}{\partial t_f} \delta t_f \right] = 0$$

Substituting into this equation the expressions for $\delta \lambda_j$, $j = 1, \ldots, n$, from equations (111) and (112), the following result is obtained:

$$\frac{\partial P}{\partial t_f} + H_f + \sum_{q=a+1}^{n} \frac{\partial^2 P}{\partial t_f \partial x_{q_f}} \left(\delta x_{q_f} + g_{q_f} \delta t_f\right)$$

$$+ \frac{\partial^2 P}{\partial t_f^{\,2}} \delta t_f + \left(\sum_{j=1}^{n} \sum_{i=a+1}^{n} \frac{\partial P}{\partial x_{i_f}} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta t_f \right.$$

$$\left. + \sum_{i=1}^{a} \bar{\mu}_i \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta t_f + \delta \mu_j - \frac{\partial H}{\partial x_j} \delta t_f \right) g_j$$

$$+ \sum_{i=a+1}^{n} \left[\sum_{i=a+1}^{n} \frac{\partial P}{\partial x_{i_f}} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta t_f + \sum_{i=1}^{a} \bar{\mu}_i \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta t_f \right.$$

$$+ \sum_{i=a+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial x_{j_f}} \left(\delta x_{i_f} + g_{i_f} \delta t_f\right) + \frac{\partial^2 P}{\partial x_{j_f} \partial t_f} \delta t_f$$

$$\left. - \frac{\partial H}{\partial x_i} \delta t_f \right] g_j + \sum_{i=1}^{n} \lambda_i \left[\sum_{q=1}^{n} \frac{\partial g_i}{\partial x_{q_f}} \left(\delta x_{q_f} + g_{q_f} \delta t_f\right) \right.$$

$$\left. + \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_{k_f}} \left(\delta y_{k_f} + \dot{y}_{k_f} \delta t_f\right) + \frac{\partial g_i}{\partial t_f} \delta t_f \right] = 0 \qquad (121)$$

Comparing equations (114) and (121), it is seen that the transversality condition for free δt_f is a linearized version of the transversality condition for free t_f for the original problem.

Comparing equations (115) and (107), it is seen that the condition for $\delta\mu$ is equal to a linearized version of the end constraints ψ_j.

15. REFINEMENT TRANSVERSALITY CONDITIONS FOR INSERTION INTO CIRCULAR ORBIT

From the discussion in the previous section, the transversality conditions are linearized versions of the transversality condition for the original problems. For the problem defined in section 8, P^* is given as follows:

$$P^* = -\frac{m_f}{m_o} + \mu_1 \left[\left(u_f^2 + v_f^2\right) - \frac{\mu}{\bar{R}}\right]$$

$$+ \mu_2\left(u_f x_f + v_f y_f\right) + \mu_3 w_f$$

$$+ \mu_4 \left[\left(x_f^2 + y_f^2\right) - \bar{R}^2\right] + \mu_5 z_f$$

The transversality conditions for x_{i_f} are given by $\frac{\partial P^*}{\partial x_{i_f}} = \lambda_{i_f}$.

Taking the indicated partials, the following partials are obtained:

$$\lambda_m\left(t_f\right) = -\frac{1}{m_o} \tag{122}$$

$$\lambda_u\left(t_f\right) = 2\mu_1 u_f + \mu_2 x_f \tag{123}$$

$$\lambda_v\left(t_f\right) = 2\mu_1 v_f + \mu_2 v_f \tag{124}$$

$$\lambda_w\left(t_f\right) = \mu_3 \tag{125}$$

$$\lambda_x\left(t_f\right) = \mu_2 u_f + 2\mu_4 x_f \tag{126}$$

$$\lambda_y\left(t_f\right) = \mu_2 v_f + 2\mu_4 y_f \tag{127}$$

and

$$\lambda_z\left(t_f\right) = \mu_5 \tag{128}$$

Equations (123), (124), (126), and (127) represent four equations with the three constant multipliers μ_1, μ_2, and μ_4. These constant multipliers can be eliminated to obtain the following single transversality condition:

$$u_f\ \lambda_{v_f} - v_f\ \lambda_{u_f} + x_f\ \lambda_{y_f} - y_f\ \lambda_{y_f} = 0$$

A linearized version of this equation is as follows:

$$\begin{aligned}
& u_f\ \lambda_{v_f} - v_f\ \lambda_{u_f} + x_f\ \lambda_{y_f} - y_f\ \lambda_{x_f} \\
& \quad + u_f\left(\delta\lambda_{v_f} + \dot{\lambda}_{v_f}\ \delta t_f\right) + \lambda_{v_f}\ \left(\delta u_f + \dot{u}_f\ \delta t_f\right) \\
& \quad - v_f\left(\delta\lambda_{u_f} + \dot{\lambda}_{u_f}\ \delta t_f\right) - \lambda_{u_f}\ \left(\delta v_f + \dot{v}_f\ \delta t_f\right) \\
& \quad + x_f\left(\delta\lambda_{y_f} + \dot{\lambda}_{y_f}\ \delta t_f\right) + \lambda_{y_f}\ \left(\delta x_f + \dot{x}_f\ \delta t_f\right) \\
& \quad - y_f\left(\delta\lambda_{x_f} + \dot{\lambda}_{x_f}\ \delta t_f\right) - \lambda_{x_f}\ \left(\delta y_f + \dot{y}_f\ \delta t_f\right) = 0
\end{aligned} \tag{129}$$

A linearized version of equation (122) is given by

$$\lambda_m\left(t_f\right) + \delta\lambda_m + \dot{\lambda}_m\ \delta t_f = -\frac{1}{m_o} \tag{130}$$

Linearized versions of the constraint equations are as follows:

$$\left(u_f^2 + v_f^2\right) - \frac{\mu}{\bar{R}} + 2u_f\left(\delta u_f + \dot{u}_f\,\delta t_f\right) + 2v_f\left(\delta v_f + \dot{v}_f\,\delta t_f\right) = 0 \tag{131}$$

$$u_f x_f + v_f y_f + u_f\left(\delta x_f + \dot{x}_f\,\delta t_f\right) + x_f\left(\delta u_f + \dot{u}_f\,\delta t_f\right) + v_f\left(\delta y_f + \dot{y}_f\,\delta t_f\right) + y_f\left(\delta v_f + \dot{v}_f\,\delta t_f\right) = 0 \tag{132}$$

$$\left(x_f^2 + y_f^2\right) - \bar{R}^2 + 2x_f\left(\delta x_f + \dot{x}_f\,\delta t_f\right) + 2y_f\left(\delta y_f + \dot{y}_f\,\delta t_f\right) = 0 \tag{133}$$

$$z_f + \left(\delta z_f + \dot{z}_f\,\delta t_f\right) = 0 \tag{134}$$

and

$$w_f + \left(\delta w_f + \dot{w}_f\,\delta t_f\right) = 0 \tag{135}$$

A linearized version of the transversality condition

$$H_f + \frac{\partial P}{\partial t_f} = 0$$

for free t_f is given by

$$H_f + \sum_{j=1}^{7} \frac{\partial H}{\partial x_{j_f}} \Delta x_{j_f} + \frac{\partial H}{\partial \alpha}\Delta\alpha + \frac{\partial H}{\partial \theta}\Delta\theta + \sum_{j=1}^{7} \frac{\partial H_f}{\partial \lambda_{j_f}} \Delta\lambda_{j_f} = 0$$

Since

$$\frac{\partial H}{\partial x_{j_f}} = \sum_{i=1}^{7} \lambda_i \frac{\partial g_i}{\partial x_j} = -\dot{\lambda}_{j_f}$$

and

$$\frac{\partial H_f}{\partial \lambda_{j_f}} = -g_j$$

the above equation reduces to

$$H_f - \sum_{j=1}^{7} \dot{\lambda}_{j_f} \delta x_{j_f} + \left(\frac{\partial H}{\partial \alpha}\right)_f \Delta\alpha_f + \left(\frac{\partial H}{\partial \theta}\right)_f \Delta\theta_f + \sum_{j=1}^{7} g_{j_f} \delta\lambda_{j_f} = 0$$

(136)

where

$$\frac{\partial H}{\partial \alpha} = \sum_{i=1}^{7} \lambda_i \frac{\partial g_i}{\partial \alpha}$$

and

$$\frac{\partial H}{\partial \theta} = \sum_{i=1}^{7} \lambda_i \frac{\partial g_i}{\partial \theta}$$

16. METHOD OF SOLUTION FOR REFINEMENT SCHEME

A step-by-step method of solution for the penalty function process is given in section 13. For the first cycle in the refinement method the $\alpha(t)$, $\theta(t)$, t_1, t_2, and $\lambda_i(t_o)$, $i = 1, \ldots, 7$ values from the last cycle of the penalty process are used. The method of solution for the refinement method is as follows:

Step. 1. Calculate the transition matrix using the same method as described in step 5 of the penalty scheme.

Step 2. Equations (129) through (136), equation (56) evaluated at t_1 and t_2, and the set of equations given by (105) and (106) represent a set of 24 equations and 24 unknowns. The 24 unknowns and the method of solution are the same as given in step 6 of the penalty scheme.

Step 3. This step is the same as step 7 in section 13.

Step 4. The values of δt_1, δt_2, $\delta\lambda_i(t_f)$, $i = 1, \ldots, 7$ calculated in step 2, and the values of $\delta\alpha(t)$ and $\delta\theta(t)$ calculated in step 3, are added to the values of t_1, t_2, $\alpha(t)$, $\theta(t)$, $\lambda_i(t_o)$, $i = 1, \ldots, 7$ for reference trajectory.

The process is repeated in steps 1 through 4 until all of the following conditions are satisfied within reasonable tolerances:

$$\left(\frac{\partial H}{\partial T}\right)_{t_1} = 0$$

$$\left(\frac{\partial H}{\partial T}\right)_{t_2} = 0$$

$$\frac{\partial H}{\partial \alpha}(t) = 0$$

$$\frac{\partial H}{\partial \theta}(t) = 0$$

$$\theta(t) = \theta_{min}(t)$$

$$\alpha(t) = \alpha_{min}(t)$$

$$H(t_f) = 0$$

$$\lambda_m(t_f) = -\frac{1}{m_o}$$

$$u_f \lambda_{v_f} - v_f \lambda_{u_f} + x_f \lambda_{y_f} - y_f \lambda_{x_f} = 0$$

$$\left(u_f^2 + v_f^2\right) - \frac{\mu}{\bar{R}} = 0$$

$$\left(u_f\, x_f + v_f\, y_f\right) = 0$$

$$w_f = 0$$

$$\left(x_f^{\,2} + y_f^{\,2}\right) - \bar{R}^2 = 0$$

and

$$z_f = 0$$

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